

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 7.2—Solving Linear Recurrence Relations



— Page references correspond to locations of Extra Examples icons in the textbook.

p.463, icon at Example 3

#1. Solve: $a_n = 2a_{n-1} + 3a_{n-2}$, $a_0 = 0$, $a_1 = 1$.

Solution:

Using $a_n = r^n$, the following characteristic equation is obtained:

$$r^2 - 2r - 3 = 0$$

The left side factors as $(r - 3)(r + 1)$, yielding the roots 3 and -1 . Hence, the general solution to the given recurrence relation is

$$a_n = c3^n + d(-1)^n.$$

Using the initial conditions $a_0 = 0$ and $a_1 = 1$ yields the system of equations

$$c + d = 0$$

$$3c - d = 1$$

with solution $c = 1/4$ and $d = -1/4$. Therefore, the solution to the given recurrence relation is

$$a_n = \frac{1}{4} \cdot 3^n - \frac{1}{4} \cdot (-1)^n.$$

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#2. Solve: $a_n = -7a_{n-1} - 10a_{n-2}$, $a_0 = 3$, $a_1 = 3$.

Solution:

Using $a_n = r^n$ yields the characteristic equation $r^2 + 7r + 10 = 0$, or $(r + 5)(r + 2) = 0$. Therefore the general solution is

$$a_n = c(-5)^n + d(-2)^n.$$

The initial conditions give the system of equations

$$c + d = 3$$

$$-5c - 2d = 3.$$

The solution to the system is $c = -3$ and $d = 6$. Hence, the solution to the recurrence relation is

$$a_n = (-3)(-5)^n + 6(-2)^n.$$

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#3. Solve: $a_n = 10a_{n-1} - 25a_{n-2}$, $a_0 = 3$, $a_1 = 4$.

Solution:

Using $a_n = r^n$ yields the characteristic equation $r^2 - 10r + 25 = 0$, or $(r - 5)(r - 5) = 0$, with 5 as a repeated solution. Therefore the general solution is

$$a_n = c \cdot 5^n + d \cdot n \cdot 5^n.$$

The initial conditions give the system of equations

$$\begin{aligned} c &= 3 \\ 5c + 5d &= 4. \end{aligned}$$

The solution to the system is $c = 3$ and $d = -11/5$. Hence, the solution to the recurrence relation is

$$a_n = 3 \cdot 5^n - \frac{11}{5} \cdot n \cdot 5^n.$$

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#4. Suppose that the characteristic equation of a linear homogeneous recurrence relation with constant coefficients is

$$(r - 3)^4(r - 2)^3(r + 6) = 0.$$

Write the general solution of the recurrence relation.

Solution:

$$a_n = a3^n + bn3^n + cn^23^n + dn^33^n + e2^n + fn2^n + gn^22^n + h(-6)^n.$$

p.469, icon at Example 11

#1. Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$, with initial condition $a_0 = 2$.

Solution:

The characteristic equation for the associated homogeneous recurrence relation is $r - 3 = 0$, which has solution $r = 3$. Therefore the general solution to the associated homogeneous recurrence relation is

$$a_n = a3^n.$$

To obtain a particular solution to the given recurrence relation, try $a_n^{(p)} = c2^n$, obtaining $c2^n = 3c2^{n-1} + 2^n$, which yields $c = -2$. Therefore a particular solution is

$$a_n^{(p)} = -2^{n+1}.$$

Hence, the general solution to the given recurrence relation is

$$a_n = a3^n - 2^{n+1}.$$

The initial condition $a_0 = 2$ gives $2 = a \cdot 1 - 2$, or $a = 4$. Therefore the solution to the given nonhomogeneous recurrence relation is

$$a_n = 4 \cdot 3^n - 2^{n+1}.$$

p.469, icon at Example 11

#2. Solve the recurrence relation $a_n = 8a_{n-1} - 12a_{n-2} + 3n$, with initial conditions $a_0 = 1$ and $a_1 = 5$.

Solution:

The characteristic equation for the associated homogeneous recurrence relation is $r^2 - 8r + 12 = 0$, which has solutions $r = 6$ and $r = 2$. Therefore, the general solution to the associated homogeneous recurrence relation is $a_n = a \cdot 6^n + b \cdot 2^n$. To obtain a particular solution to the given recurrence relation, try $a_n^{(p)} = cn + d$, obtaining

$$cn + d = 8[c(n-1) + d] - 12[c(n-2) + d] + 3n,$$

which can be rewritten as

$$n(c - 8c + 12c - 3) + (d + 8c - 8d - 24c + 12d) = 0.$$

The coefficient of n -term and the constant term must each equal 0. Therefore, we have

$$\begin{aligned}c - 8c + 12c - 3 &= 0 \\d + 8c - 8d - 24c + 12d &= 0,\end{aligned}$$

or $c = 3/5$ and $d = 48/25$.

Therefore,

$$a_n = a 6^n + b 2^n + \frac{3}{5}n + \frac{48}{25}.$$

Using the two initial conditions, $a_0 = 1$ and $a_1 = 5$, yields the system of equations

$$\begin{aligned}a6^0 + b2^0 + \frac{3}{5} \cdot 0 + \frac{48}{25} &= 1 \\a6^1 + b2^1 + \frac{3}{5} \cdot 1 + \frac{48}{25} &= 5\end{aligned}$$

and the solution is found to be $a = 27/25$ and $b = -2$. Therefore, the solution to the given recurrence relation is

$$a_n = \frac{27}{25}6^n - 2^{n+1} + \frac{3}{5}n + \frac{48}{25}.$$