3.3.4 Beta Distribution

The beta(α, β) pdf is

$$f(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, \quad \alpha > 0, \quad \beta > 0,$$

where $B(\alpha, \beta)$ denotes the beta function,

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

For $n > -\alpha$, we have

$$\begin{aligned} EX^n &= \frac{1}{B(\alpha, \beta)} \int_0^1 x^n x^{\alpha-1}(1-x)^{\beta-1} dx \\ &= \frac{B(\alpha+n, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)}. \end{aligned}$$

Then mean and variance are

$$EX = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \mathrm{Var}X = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Genesis: Handout 1.

3.3.5 Cauchy Distribution

The Cauchy distribution is a symmetric, bell-shaped distribution on $(-\infty, \infty)$ with pdf

$$f(x|\theta) = \frac{1}{\pi} \frac{1}{1+(x-\theta)^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

The mean of Cauchy distribution does not exist, that is,

$$E|X| = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{|x|}{1+(x-\theta)^2} dx = \infty.$$

Since $E|X| = \infty$, it follows that no moments of the Cauchy distribution exist. In particular, the mgf does not exist.

Genesis: Handout 2:

3.3.6 Lognormal Distribution

If X is a random variable whose logarithm is normally distributed, then X has a lognormal distribution. The pdf of X can be obtained by straightforward transformation of the normal pdf, yielding

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} e^{-(\log x - \mu)^2/(2\sigma^2)}, \quad 0 < x < \infty, \quad -\infty < \mu < \infty, \sigma > 0,$$

for the lognormal pdf.

$$EX = Ee^{\log X} = Ee^{Y} = e^{\mu + (\sigma^2/2)}.$$

$$\mathrm{Var} X = e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2}.$$

Genesis: Handout 3.