UCLA STAT 110 A

Applied Probability & Statistics for Engineers

•**Instructor**: **Ivo Dinov**,
Asst. Prof. In Statistics and Neurology

•**Teaching Assistant:** Neda Farzinnia, UCLA Statistics

University of California, Los Angeles, Spring 2004
http://www.stat.ucla.edu/~dinov/

Chapter 5

Joint Probability Distributions and Random Samples

5.1

Jointly Distributed Random Variables

Joint Probability Mass Function

Let X and Y be two discrete rv's defined on the sample space of an experiment. The *joint probability mass function* $p(x, y)$ is defined for each pair of numbers (x, y) by

$$p(x,y) = P(X = x \text{ and } Y = y)$$

Let A be the set consisting of pairs of (x, y) values, then

$$P\left[(X,Y) \in A\right] = \sum_{(x,y)\in A}\sum p(x,y)$$

Marginal Probability Mass Functions

The *marginal probability mass functions* of X and Y, denoted $p_X(x)$ and $p_Y(y)$ are given by

$$p_X(x) = \sum_y p(x,y) \qquad p_Y(y) = \sum_x p(x,y)$$

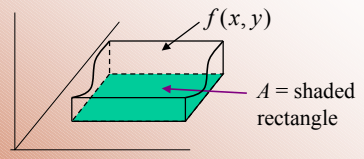
Joint Probability Density Function

Let X and Y be continuous rv's. Then $f(x, y)$ is a *joint probability density function* for X and Y if for any two-dimensional set A

$$P\left[(X,Y) \in A\right] = \iint_A f(x,y)\,dx\,dy$$

If A is the two-dimensional rectangle $\{(x,y) : a \le x \le b, c \le y \le d\}$,

$$P\left[(X,Y) \in A\right] = \int_a^b\int_c^d f(x,y)\,dy\,dx$$



$P\left[(X,Y) \in A\right]$

= Volume under density surface above A

Marginal Probability Density Functions

The *marginal probability density functions* of X and Y, denoted $f_X(x)$ and $f_Y(y)$, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \quad \text{for } -\infty < y < \infty$$

Independent Random Variables

Two random variables X and Y are said to be *independent* if for every pair of x and y values

$$p(x,y) = p_X(x) \cdot p_Y(y)$$

when X and Y are discrete or

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

when X and Y are continuous. If the conditions are not satisfied for all (x, y) then X and Y are *dependent.*

More Than Two Random Variables

If $X_1, X_2, \ldots, X_n$ are all discrete random variables, the joint pmf of the variables is the function

$$p(x_1,\ldots,x_n) = P(X_1 = x_1,\ldots,X_n = x_n)$$

If the variables are continuous, the joint pdf is the function f such that for any n intervals $[a_1,b_1]$, $\ldots,[a_n,b_n]$, $P(a_1 \le X_1 \le b_1,\ldots,a_n \le X_n \le b_n)$

$$= \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(x_1,\ldots,x_n)dx_n \ldots dx_1$$

Independence – More Than Two Random Variables

The random variables $X_1, X_2, \ldots, X_n$ are *independent* if for every subset $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ of the variables, the joint pmf or pdf of the subset is equal to the product of the marginal pmf's or pdf's.

Conditional Probability Function

Let X and Y be two continuous rv's with joint pdf $f(x, y)$ and marginal X pdf $f_X(x)$. Then for any X value x for which $f_X(x) > 0$, *the conditional probability density function of Y given that X = x* is

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} \quad -\infty < y < \infty$$

If X and Y are discrete, replacing pdf's by pmf's gives the *conditional probability mass function of Y when X = x.*

Marginal probability distributions (Cont.)

- If X and Y are discrete random variables with joint probability mass function $f_{XY}(x,y)$, then the marginal probability mass function of X and Y are

$$f_X(x) = P(X = x) = \sum_{R_x} f_{XY}(X,Y)$$

$$f_Y(y) = P(Y = y) = \sum_{Ry} f_{XY}(X,Y)$$

where R_x denotes the set of all points in the range of (X, Y) for which $X = x$ and Ry denotes the set of all points in the range of (X, Y) for which $Y = y$

Mean and Variance

- If the marginal probability distribution of X has the probability function f(x), then

$$E(X) = \mu_X = \sum_x x f_X(x) = \sum_x x\left(\sum_{R_x} f_{XY}(x,y)\right) = \sum_x \sum_{R_x} x f_{XY}(x,y)$$

$$= \sum_R x f_{XY}(x,y)$$

$$V(X) = \sigma^2{}_X = \sum_x (x-\mu_X)^2 f_X(x) = \sum_x (x-\mu_X)^2 \sum_{R_x} f_{XY}(x,y)$$

$$= \sum_x \sum_{R_x} (x-\mu_X)^2 f_{XY}(x,y) = \sum_R (x-\mu_X)^2 f_{XY}(x,y)$$

- R = Set of all points in the range of (X,Y).
- Example 5-4.

Joint probability mass function – example

The joint density, P{***X,Y***}, of the number of minutes waiting to catch the first fish, ***X***, and the number of minutes waiting to catch the second fish, ***Y***, is given below.

P {***X*** = i,***Y*** = k }		k 1	2	3	Row Sum P{ ***X*** = i }
i	1	0.01	0.02	0.08	0.11
	2	0.01	0.02	0.08	0.11
	3	0.07	0.08	0.63	0.78
Column Sum P {***Y*** =k }		0.09	0.12	0.79	1.00

- The (joint) chance of **waiting 3 minutes to catch the first fish and 3 minutes to catch the second fish** is:
- The (marginal) chance of **waiting 3 minutes to catch the first fish** is:
- The (marginal) chance of **waiting 2 minutes to catch the first fish** is (circle all that are correct):
- The chance of **waiting at least two minutes to catch the first fish** is (circle none, one or more):
- The chance of **waiting at most two minutes to catch the first fish and at most two minutes to catch the second fish** is (circle none, one or more):

Conditional probability

- Given discrete random variables X and Y with joint probability mass function $f_{XY}(X,Y)$, the conditional probability mass function of Y given X=x is

$f_{Y|x}(y|x) = f_{Y|x}(y) = f_{XY}(x,y)/f_X(x)$ for $f_X(x) > 0$

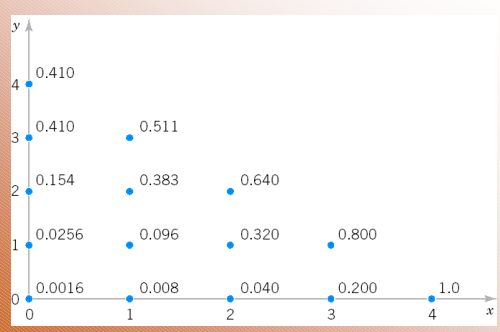
Conditional probability (Cont.)

- Because a conditional probability mass function $f_{Y|x}(y)$ is a probability mass function for all y in R_x, the following properties are satisfied:

(1) $f_{Y|x}(y) \geq 0$

(2) $\sum_{R_x} f_{Y|x}(y) = 1$

(3) $P(Y=y|X=x) = f_{Y|x}(y)$



Conditional probability (Cont.)

- Let R_x denote the set of all points in the range of (X,Y) for which X=x. The conditional mean of Y given X=x, denoted as E(Y|x) or $\mu_{Y|x}$, is

$$E(\mathrm{Y} \mid \mathrm{x}) = \sum_{R_x} y f_{\mathrm{Y}|\mathrm{x}}(y)$$

- And the conditional variance of Y given X=x, denoted as V(Y|x) or $\sigma^2_{Y|x}$ is

$$V(\mathrm{Y} \mid \mathrm{x}) = \sum_{R_x} (y-\mu_{\mathrm{Y}|\mathrm{x}})^2 f_{\mathrm{Y}|\mathrm{x}}(y) = \sum_{R_x} y^2 f_{\mathrm{Y}|\mathrm{x}}(y) - \mu^2_{\mathrm{Y}|\mathrm{x}}$$

Independence

- For discrete random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are independent.

(1) $f_{XY}(x,y) = f_X(x)\, f_Y(y)$ for all x and y

(2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

(3) $f_{X|y}(y) = f_X(x)$ for all x and y with $f_Y(y) > 0$

(4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y respectively.

5.2

Expected Values, Covariance, and Correlation

Expected Value

Let X and Y be jointly distributed rv's with pmf $p(x, y)$ or pdf $f(x, y)$ according to whether the variables are discrete or continuous. Then the *expected value* of a function $h(X, Y)$, denoted $E[h(X, Y)]$ or $\mu_{h(X,Y)}$

is

$$= \begin{cases} \sum_x \sum_y h(x,y) \cdot p(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) \cdot f(x,y)\,dx\,dy & \text{continuous} \end{cases}$$

Covariance

The *covariance* between two rv's X and Y is

$$\mathrm{Cov}(X,Y) = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$= \begin{cases} \sum_x \sum_y (x-\mu_X)(y-\mu_Y)\, p(x,y) & \text{discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y)\, f(x,y)\,dx\,dy & \text{continuous} \end{cases}$$

Short-cut Formula for Covariance

$$\boxed{\mathrm{Cov}(X,Y) = E(XY) - \mu_X \cdot \mu_Y}$$

Correlation

The *correlation coefficient* of X and Y, denoted by Corr(X, Y), $\rho_{X,Y}$, or just ρ, is defined by

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

Correlation Proposition

1. If a and c are either both positive or both negative, Corr($aX + b$, $cY + d$) = Corr(X, Y)
2. For any two rv's X and Y, $-1 \le \mathrm{Corr}(X,Y) \le 1$.

Correlation Proposition

1. If X and Y are independent, then $\rho = 0$, but $\rho = 0$ does not imply independence.
2. $\rho = 1$ or -1 iff $Y = aX + b$ for some numbers a and b with $a \neq 0$.

5.3

Statistics and their Distributions

Statistic

A *statistic* is any quantity whose value can be calculated from sample data. Prior to obtaining data, there is uncertainty as to what value of any particular statistic will result. A statistic is a random variable denoted by an uppercase letter; a lowercase letter is used to represent the calculated or observed value of the statistic.

Random Samples

The rv's $X_1,\ldots,X_n$ are said to form a (simple *random sample* of size n if

1. The X_i's are independent rv's.
2. Every X_i has the same probability distribution.

Simulation Experiments

The following characteristics must be specified:

1. The statistic of interest.
2. The population distribution.
3. The sample size n.
4. The number of replications k.

5.4

The Distribution of the Sample Mean

Using the Sample Mean

Let $X_1,\ldots, X_n$ be a random sample from a distribution with mean value μ and standard deviation σ. Then

1. $E\left(\overline{X}\right) = \mu_{\overline{X}} = \mu$
2. $V\left(\overline{X}\right) = \sigma^2_{\overline{X}} = \sigma^2 / n$

In addition, with $T_o = X_1 + \ldots + X_n$, $E\left(T_o\right) = n\mu$, $V\left(T_o\right) = n\sigma^2$, and $\sigma_{T_o} = \sqrt{n}\sigma$.

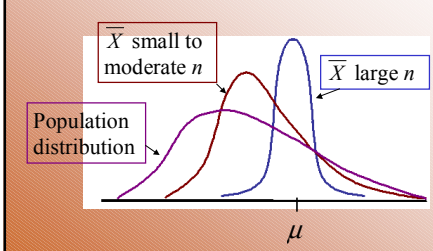
Normal Population Distribution

Let $X_1,\ldots, X_n$ be a random sample from a normal distribution with mean value μ and standard deviation σ. Then for any n, $\overline{X}$ is normally distributed, as is T_o.

The Central Limit Theorem

Let $X_1,\ldots, X_n$ be a random sample from a distribution with mean value μ and variance σ^2. Then if n sufficiently large, $\overline{X}$ has approximately a normal distribution with $\mu_{\overline{X}} = \mu$ and $\sigma^2_{\overline{X}} = \sigma^2 / n$, and T_o also has approximately a normal distribution with $\mu_{T_o} = n\mu$, $\sigma_{T_o} = n\sigma^2$. The larger the value of n, the better the approximation.

The Central Limit Theorem



Rule of Thumb

If $n > 30$, the Central Limit Theorem can be used.

Approximate Lognormal Distribution

Let $X_1,\ldots, X_n$ be a random sample from a distribution for which only positive values are possible $[P(Xi > 0) = 1]$. Then if n is sufficiently large, the product $Y = X_1X_2\ldots X_n$ has approximately a lognormal distribution.

5.5

The Distribution of a Linear Combination

Linear Combination

Given a collection of n random variables $X_1,\ldots, X_n$ and n numerical constants $a_1,\ldots,a_n$, the rv

$$Y = a_1X_1 + \ldots + a_nX_n = \sum_{i=1}^{n} a_i X_i$$

is called a *linear combination* of the X_i's.

Expected Value of a Linear Combination

Let $X_1,\ldots, X_n$ have mean values $\mu_1, \mu_2, \ldots, \mu_n$ and variances of $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively

Whether or not the X_i's are independent,

$$E(a_1X_1 + \ldots + a_nX_n) = a_1E(X_1) + \ldots + a_nE(X_n)$$
$$= a_1\mu_1 + \ldots + a_n\mu_n$$

Variance of a Linear Combination

If $X_1,\ldots, X_n$ are independent,

$$V(a_1X_1 + \ldots + a_nX_n) = a_1^2V(X_1) + \ldots + a_n^2V(X_n)$$
$$= a_1^2\sigma_1^2 + \ldots + a_n^2\sigma_n^2$$

and

$$\sigma_{a_1X_1+\ldots+a_nX_n} = \sqrt{a_1^2\sigma_1^2 + \ldots + a_n^2\sigma_n^2}$$

Variance of a Linear Combination

For any $X_1, \ldots, X_n$,

$$V\left(a_1X_1 + \ldots + a_nX_n\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{Cov}\left(X_i, X_j\right)$$

Difference Between Two Random Variables

$$E\left(X_1 - X_2\right) = E\left(X_1\right) - E\left(X_2\right)$$

and, if X_1 and X_2 are independent,

$$V\left(X_1 - X_2\right) = V\left(X_1\right) + V\left(X_2\right)$$

Difference Between Normal Random Variables

If $X_1, X_2, \ldots X_n$ are independent, normally distributed rv's, then any linear combination of the X_i's also has a normal distribution. The difference $X_1 - X_2$ between two independent, normally distributed variables is itself normally distributed.

Central Limit Theorem – heuristic formulation

Central Limit Theorem:

When sampling from almost any distribution,

$\overline{X}$ is approximately Normally distributed in large samples.

Show Sampling Distribution Simulation Applet:
file:///C:/Ivo.dir/UCLA_Classes/Winter2002/AdditionalInstructorAids/
SamplingDistributionApplet.html

Independence

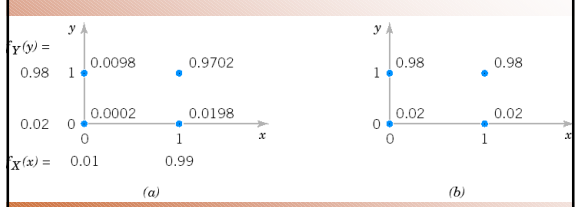
- For discrete random variables X and Y, if any one of the following properties is true, the others are also true, and X and Y are independent.

  (1) $f_{XY}(x,y) = f_X(x)\, f_Y(y)$ for all x and y

  (2) $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$

  (3) $f_{X|y}(y) = f_X(x)$ for all x and y with $f_Y(y) > 0$

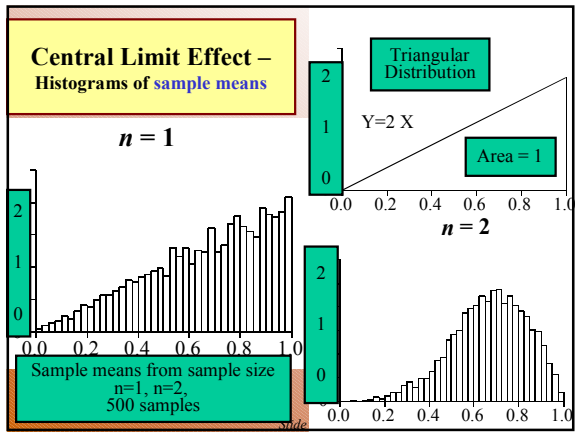
  (4) $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y respectively.



Recall we looked at the sampling distribution of $\overline{X}$

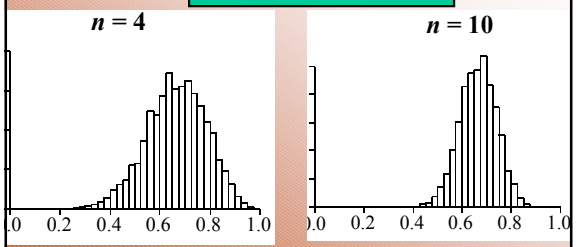
- For the sample mean calculated from a random sample, $E(\overline{X}) = \mu$ and $SD(\overline{X}) = \sigma/\sqrt{n}$, provided $\overline{X} = (X_1+X_2+ \ldots + X_n)/n$, and $X_k \sim N(\mu, \sigma)$. Then
- $\overline{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$. And variability from sample to sample in the ***sample-means*** is given by the variability of the individual observations divided by the square root of the sample-size. In a way, averaging decreases variability.

Central Limit Effect –
Histograms of sample means

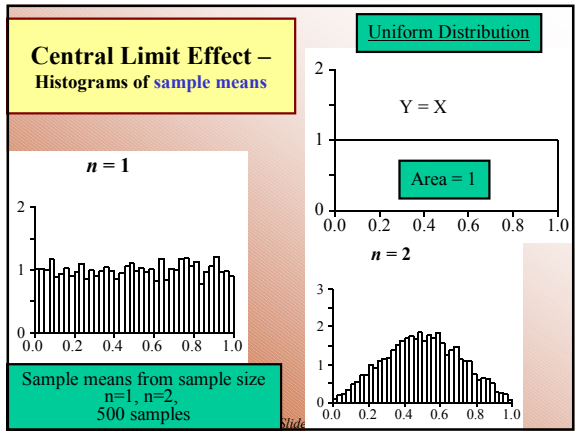


Central Limit Effect -- Histograms of sample means

Triangular Distribution
Sample sizes n=4, n=10

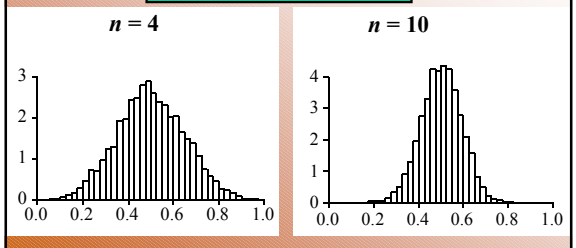


Central Limit Effect –
Histograms of sample means

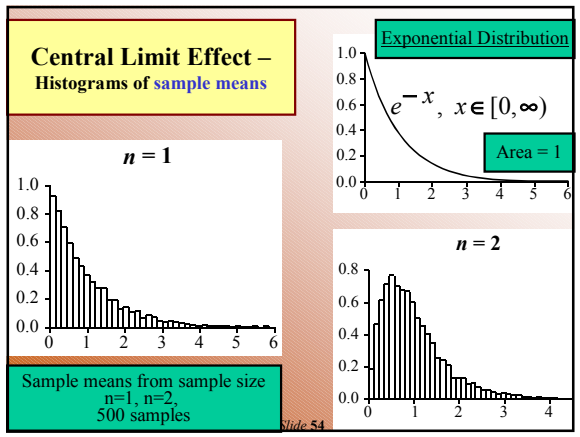


Central Limit Effect -- Histograms of sample means

Uniform Distribution
Sample sizes n=4, n=10

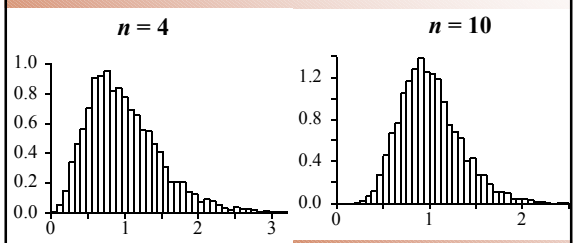


Central Limit Effect –
Histograms of sample means



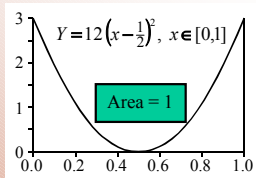
Central Limit Effect -- Histograms of sample means

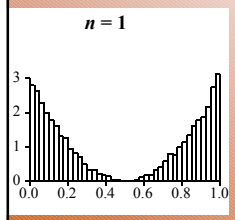
Exponential Distribution
Sample sizes n=4, n=10

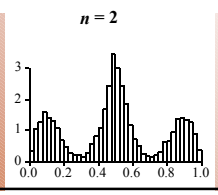


Central Limit Effect – Histograms of sample means

Quadratic U Distribution



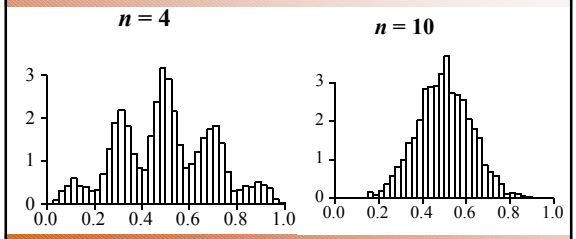




Sample means from sample size n=1, n=2, 500 samples

Central Limit Effect -- Histograms of sample means

Quadratic U Distribution
Sample sizes n=4, n=10



Central Limit Theorem – heuristic formulation

Central Limit Theorem:
When sampling from almost any distribution,
$\overline{X}$ is approximately Normally distributed in large samples.

Show Sampling Distribution Simulation Applet:
file:///C:/Ivo.dir/UCLA_Classes/Winter2002/AdditionalInstructorAids/
SamplingDistributionApplet.html

Central Limit Theorem – theoretical formulation

Let $\{X_1, X_2, \ldots, X_k, \ldots\}$ be a sequence of independent observations from one specific random process. Let and $E(X) = \mu$ and $SD(X) = \sigma$ and both be finite ($0 < \sigma < \infty$; $|\mu| < \infty$). If $\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$, sample-avg,

Then $\overline{X}$ has a distribution which approaches $N(\mu, \sigma^2/n)$, as $n \to \infty$.