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103
RL, RC, and RLC Circuits

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Circuits that contain only resistive elements can be described by a set of algebraic equations that are obtained by systematically applying Kirchhoff's current and voltage laws to the circuit. When a circuit contains energy storage elements—that is, inductors and capacitors—Kirchhoff's laws are still valid, but their application leads to a differential equation (DE) model of the circuit instead of an algebraic model. A DE model can be solved by classical DE methods, by time-domain convolution, and by Laplace transform methods.

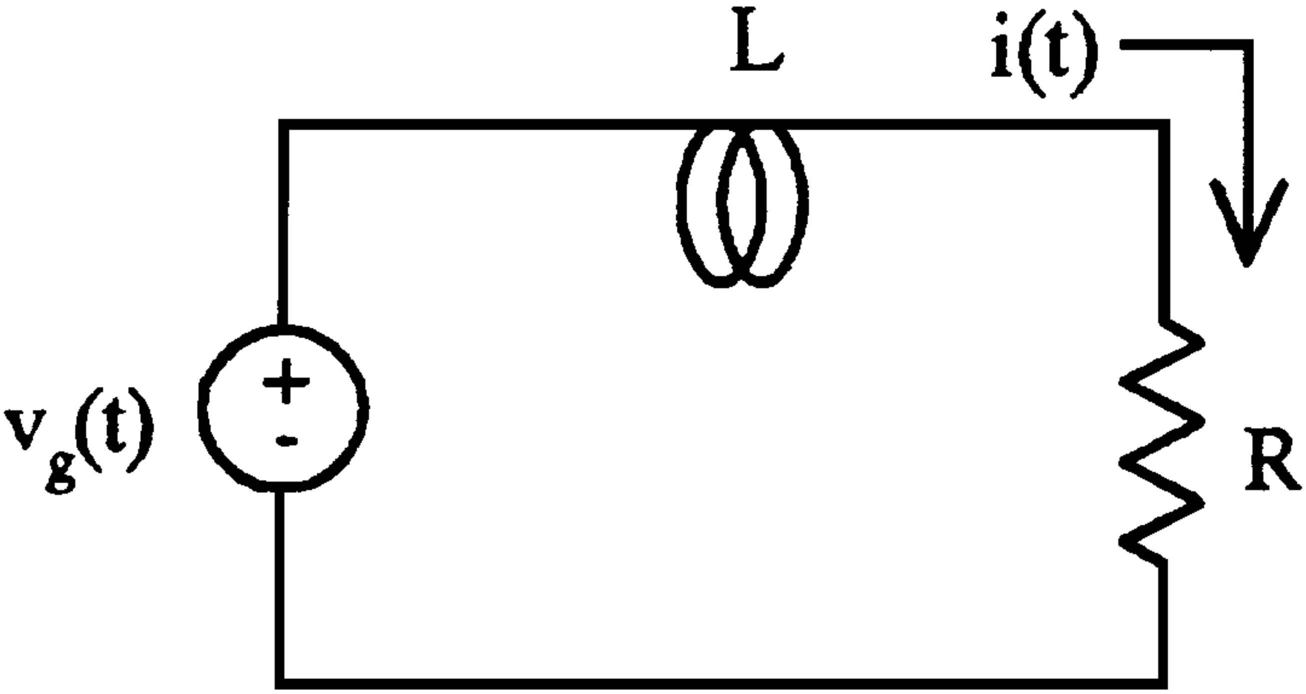
103.1 RL Circuits

A series RL circuit can be analyzed by applying Kirchhoff's voltage law to the single loop that contains the elements of the circuit. For example, writing Kirchhoff's voltage law for the circuit in Fig. 103.1 leads to the following first-order DE stating that the voltage drop across the inductor and the resistor must equal the voltage drop across the voltage source:

$$L\frac{di}{dt} + i(t)R = v_g(t) \qquad (103.1)$$

This DE belongs to a family of linear, constant-coefficient, ordinary differential equations. Various approaches can be taken to find the solution for $i(t)$, the current through the elements of the circuit. The first is to solve the DE by classical (time-domain) methods; the second is to solve the equation by using Laplace transform theory. [A third approach, waveform convolution, will not be considered here (see Ziemer *et al.,* 1993).]

Figure 103.1 Series RL circuit.



The complete time-domain solution to the DE is formed as the sum of two parts, called the **natural solution** and the **particular solution**:

$$i(t) = i_N(t) + i_P(t) \qquad (103.2)$$

The natural solution, $i_N(t)$, is obtained by solving the (homogeneous) DE with the sources (or forcing functions) turned off. The circuit in this example has the following homogeneous DE:

$$L\frac{di}{dt} + i(t)R = 0 \qquad (103.3)$$

The solution to this equation is the exponential form: $i_N(t) = Ke^{st}$, where *K* is an arbitrary constant. To verify this solution, substitute Ke^{st} into Eq. (103.3) and form

$$LKse^{st} + RKe^{st} = 0 \qquad (103.4)$$

Rearranging gives:

$$[SL + R]Ke^{st} = 0 \qquad (103.5)$$

The factor Ke^{st} can be canceled because the factor e^{st} is nonzero, and the constant *K* must be nonzero for the solution to be nontrivial. This reduction leads to the characteristic equation

$$sL + R = 0 \qquad (103.6)$$

The **characteristic equation** of this circuit specifies the value of *s* for which $i_N(t) = Ke^{st}$ solves Eq. (103.3). By inspection, $s = -R/L$ solves Eq. (103.6), and $i_N(t) = Ke^{-t/\tau}$ satisfies Eq. (103.3), and $\tau = L/R$ is called the *time constant* of the circuit. The value $s = -R/L$ is called the **natural frequency** of the circuit.

In general, higher-order circuits (those with higher-order differential equation models and corresponding higher-order algebraic characteristic equations) will have several natural frequencies, some of which may have a complex value—that is, $s = \sigma + j\omega$ —where σ is an exponential damping factor and ω is the undamped frequency of oscillation. The natural frequencies of a circuit play a key role in governing the dynamic response of the circuit to an input by determining the form and the duration of the transient waveform of the response [Ciletti, 1988].

The particular solution of this circuit's DE model is a function $i_P(t)$ that satisfies Eq. (103.1) for a given source $v_g(t)$. For example, the constant input signal $v_g(t) = V$ has the particular solution $i_P(t) = V/R$. [This can be verified by substituting this expression into Eq. (103.1).] The complete solution to the differential equation when $v_g(t) = V$ is given by

$$i(t) = Ke^{(-R/L)t} + V/R = Ke^{-t/\tau} + V/R \qquad (103.7)$$

The sources that excite physical circuits are usually modeled as being turned off before being applied at some specific time, say t_0, and the objective is to find a solution to its DE model for $t \geq t_0$. This leads to consideration of boundary conditions that constrain the behavior of the circuit and determine the unknown parameters in the solution of its DE model. The boundary conditions of the DE model of a physical circuit are determined by the energy that is stored in the circuit when the source is initially applied. For example, the inductor current in Fig. 103.1 could have the constraint given as $i(t_0) = i_0$, where i_0 is a constant. If a circuit is modeled by a constant-coefficient DE, the time of application can be taken to be $t_0 = 0$ without any loss in generality. (Note: The physical conditions that created i_0 are not of concern.)

The value of the parameter *K* in Eq. (103.7) is specified by applying the given boundary condition to the waveform of Eq. (103.7), as follows:

$$i(0^+) = i_0 = K + V/R \qquad (103.8)$$

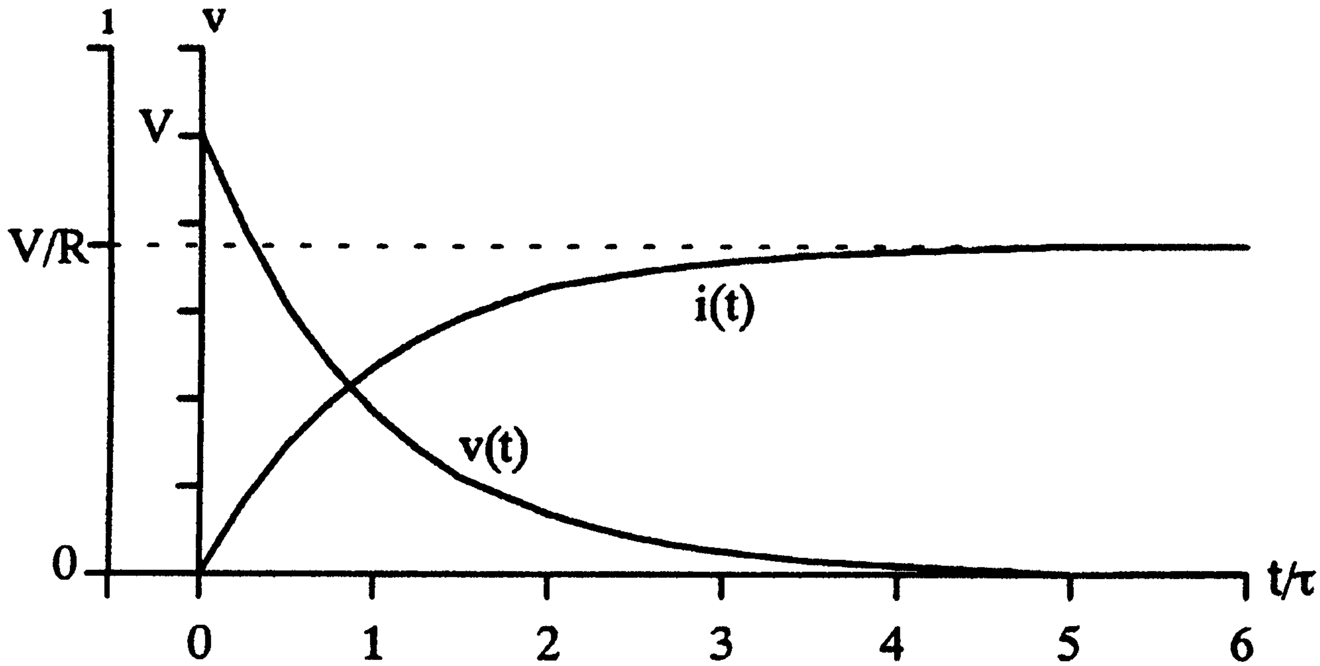
and so $K = i_0 - V/R$.

The response of the circuit to the applied input signal is the complete solution of the DE for $t \geq 0$ evaluated with coefficients that conform to the boundary conditions. In the RL circuit example the complete response to the applied step input is

$$i(t) = [i_0 - V/R]e^{-t/\tau} + V/R, \qquad t \geq 0 \qquad (103.9)$$

The waveforms of $i(t)$ and $v(t)$, the inductor voltage, are shown in Fig. 103.2 for the case in which the circuit is initially at rest, that is, $i_0 = 0$. The time axis in the plot has been normalized by τ.

Figure 103.2 Waveforms of $v(t)$ and $i(t)$ in the series RL circuit.



The physical effect of the inductor in this circuit is to provide inertia to a change in current in the series path that contains the inductor. An inductor has the physical property that its current is a continuous variable whenever the voltage applied across the terminals of the inductor has a bounded waveform (that is, no impulses). The current flow through the inductor is controlled by the voltage that is across its terminals. This voltage causes the accumulation of magnetic flux, which ultimately determines the current in the circuit. The initial voltage applied across the inductor is given by $v(0^+) = V - i(0^+)R$. If the inductor is initially relaxed—that is, $i(0^-) = 0$—the continuity property of the inductor current dictates that $i(0^+) = i(0^-) = 0$. So $v(0^+) = V$. All of the applied voltage initially appears across the inductor. When this voltage is applied for an interval of time, a magnetic flux accumulates and current is established through the inductor. Mathematically, the integration of this voltage causes the current in the circuit. The current waveform in Fig. 103.2 exhibits exponential growth from its initial value of 0 to its final (steady state) value of V/R. The inductor voltage decays from its initial value to its steady state value of 0, and the inductor appears to be a "short circuit" to the steady state DC current.

103.2 RC Circuits

A capacitor acts like a reservoir of charge, thereby preventing rapid changes in the voltage across its terminals. A capacitor has the physical property that its voltage must be a continuous variable when its current is bounded. The DE model of the parallel RC circuit in Fig. 103.3 is described according to Kirchhoff's current law by

$$C\frac{dv}{dt} + \frac{v}{R} = i_g(t) \qquad (103.10)$$

Taking the one-sided Laplace transform [Ziemer, 1993] of this differential equation gives

$$sCV(s) - Cv(0^-) + \frac{V(s)}{R} = I_g(s) \qquad (103.11)$$

where $V(s)$ and $I_g(s)$ denote the Laplace transforms of the related time-domain variables. (Note that the Laplace transform of the derivative of a variable explicitly incorporates the initial condition of the variable into the model of the circuit's behavior.) Rearranging this algebraic equation gives the following:

$$V(s) = \frac{I_g(s)}{sC + 1/R} + \frac{Cv(0)}{sC + 1/R} \qquad (103.12)$$

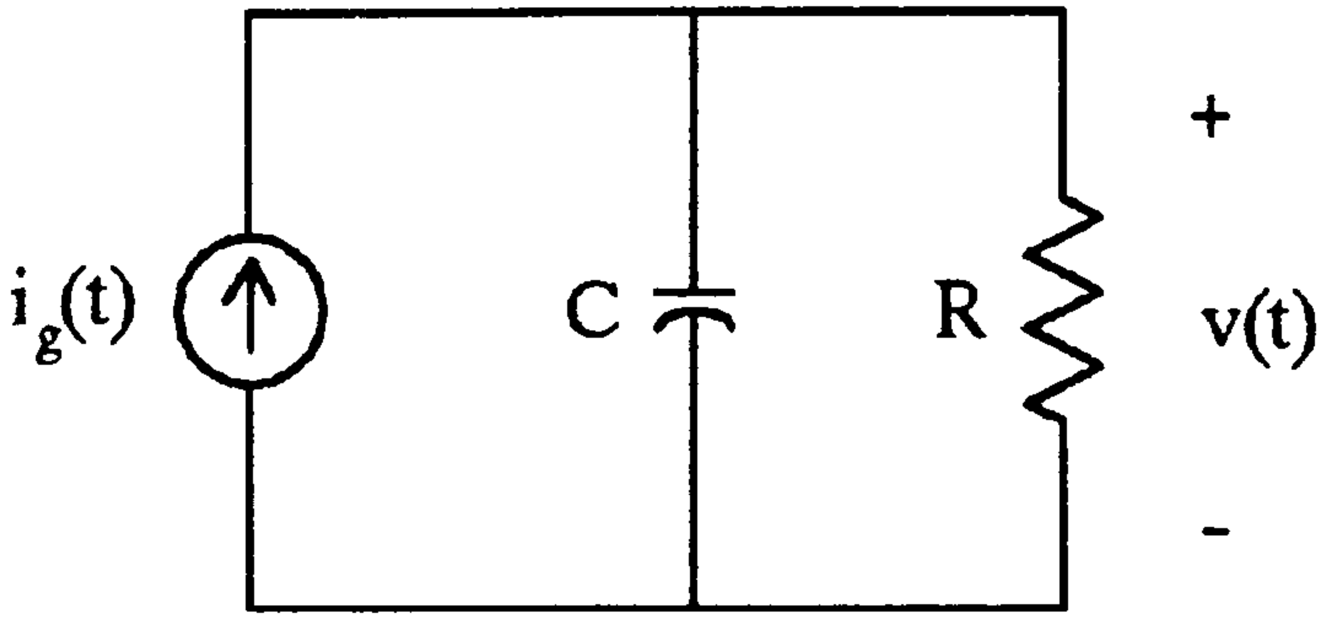
$$V(s) = I_g(s)H(s) + \frac{v(0)}{s + 1/(RC)} \qquad (103.13)$$

where the *s*-domain function

$$H(s) = \frac{1}{sC + (1/R)} = \frac{1/C}{s + (1/RC)} \qquad (103.14)$$

is called the input/output **transfer function** of the circuit. The expression in Eq. (103.13) illustrates an important property of linear circuits: the response of a linear RLC circuit variable is the superposition of the effect of the applied source and the effect of the initial energy stored in the circuit's capacitors and inductors. Another important fact is that the roots of the denominator polynomial of $H(s)$ are the natural frequencies of the circuit (assuming no cancellation between numerator and denominator factors).

Figure 103.3 Parallel RC circuit.



If a circuit is initially relaxed—that is, all capacitors and inductors are de-energized [set $v(0^-) = 0$ in Eq. (103.13)], then the transfer function defines the ratio of the Laplace transform of the circuit's response to the Laplace transform of its stimulus. Alternatively, the transfer function and the Laplace transform of the input signal determine the Laplace transform of the output signal according to the simple product

$$V(s) = I_g(s)H(s) \qquad (103.15)$$

The transfer function of a circuit can define a voltage ratio, a current ratio, a voltage-to-current ratio (impedance) or a current-to- voltage ratio (admittance). In this circuit $H(s)$ relates the output (response of the capacitor voltage) to the input (i.e., the applied current source). Thus, $H(s)$ is actually a generalized impedance function. If a circuit is initially relaxed its transfer function contains all of the information necessary to determine the response of the circuit to any given input signal.

Suppose that the circuit has an initial capacitor voltage and that the applied source in Eq. (103.13) is given by $i_g(t) = Iu(t)$, a step of height *I*. The Laplace transform [Ciletti, 1988] of the step waveform is given by $I_g(s) = I/s$, so the Laplace transform of $v(t)$, denoted by $V(s)$, is given by

$$V(s) = \frac{I/(sC)}{s + 1/(RC)} + \frac{v(0^-)}{s + 1/(RC)} \qquad (103.16)$$

This expression for $V(s)$ can be expanded algebraically into partial fractions as

$$V(s) = \frac{IR}{s} - \frac{IR}{s + 1/(RC)} + \frac{v(0^-)}{s + 1/(RC)} \qquad (103.17)$$

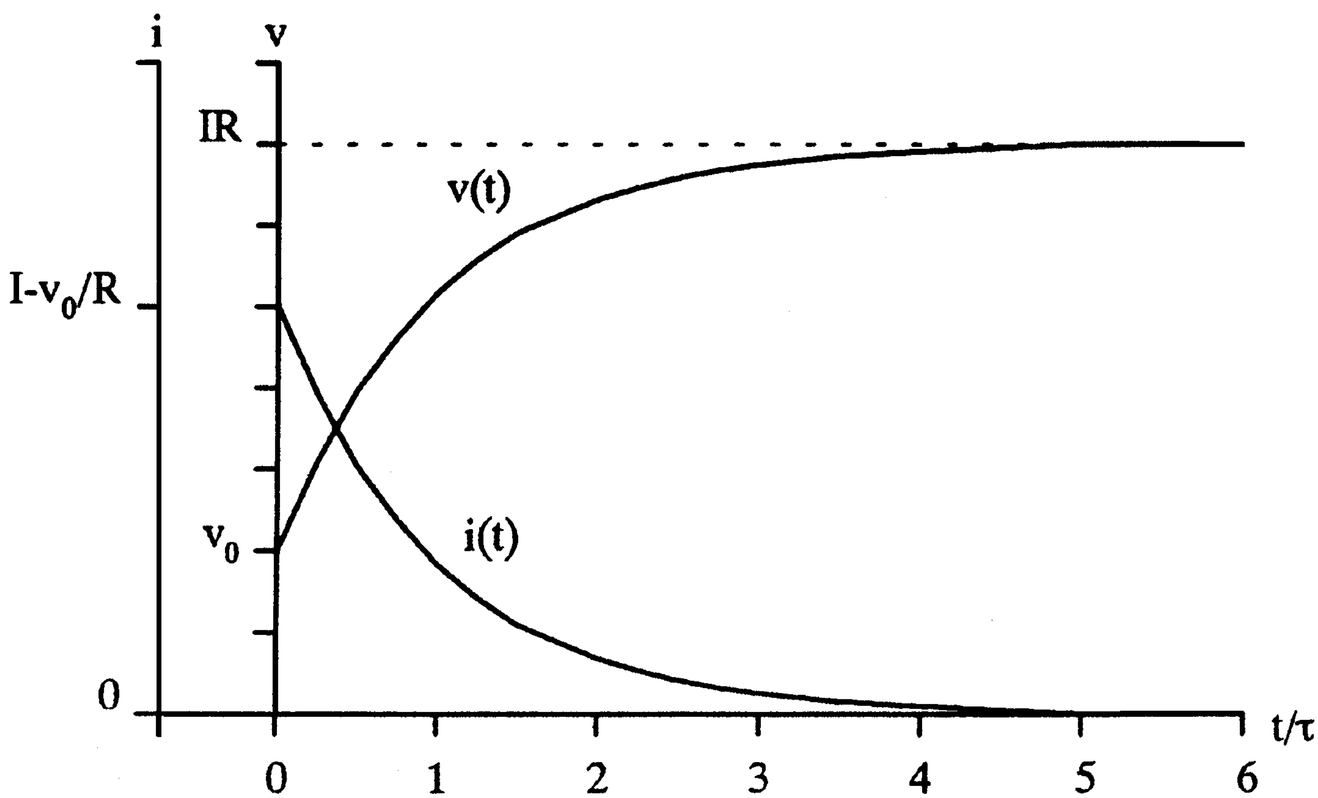
Associating the s-domain Laplace transform factor $1/(s+a)$ with the time-domain function $e^{-at}u(t)$ and taking the inverse Laplace transform of the individual terms of the expansion gives

$$v(t) = IR + [v(0^-) - IR]e^{-t/(RC)}, \qquad t \geq 0 \qquad (103.18)$$

When the source is applied to this circuit, the capacitor charges from its initial voltage to the steady state voltage given by $v(\infty) = IR$. The response of $v(t)$ is shown in Fig. 103.4 with $\tau = RC$ and $v_o = v(0^-)$. The capacitor voltage follows an exponential transition from its initial value to its steady state value at a rate determined by its time constant, τ. A similar analysis would show that the response of the capacitor current is given by

$$i(t) = [I - v(0^-)/R]e^{-t/RC}, \qquad t \geq 0 \qquad (103.19)$$

Figure 103.4 Step response of an initially relaxed RC circuit.



Capacitors behave like short circuits to sudden changes in current. The initial capacitor voltage determines the initial resistor current by Ohm's law: $v(0^+)/R$. Any initial current supplied by the source in excess of this amount will pass through the capacitor as though it were a short circuit. As the capacitor builds voltage, the resistor draws an increasing amount of current and ultimately conducts all of the current supplied by the constant source. In steady state the capacitor looks like an open circuit to the constant source—$i(\infty) = 0$—and it conducts no current.

103.3 RLC Circuits

Circuits that contain inductors and capacitors exhibit dynamical effects that combine the inductor's inertia to sudden changes in current with the capacitor's inertia to sudden changes in voltage. The topological arrangement of the components in a given circuit determines the behavior that results from the interaction of the currents and voltages associated with the individual circuit elements.

The parallel RLC circuit in Fig. 103.5(a) has an s-domain counterpart [see Fig. 103.5(b)], sometimes referred to as a *transformed circuit,* that is obtained by replacing each time-domain variable by its Laplace transform, and each physical component by a Laplace transform model of the component's voltage-current relationship. Here, for example, the physical capacitor is replaced by a model that accounts for the impedance of the capacitor and the capacitor's initial voltage. The additional sources account for the possibly nonzero initial values of capacitor voltage and inductor current. Algebraic expressions of Kirchhoff's laws are written from the Laplace model of the circuit. Applying Kirchhoff's current law to the circuit of Fig. 103.5 gives

$$sCV(s) + \frac{V(s)}{sL} + \frac{V(s)}{R} = I_g(s) + Cv(0^-) - i(0^-) \qquad (103.20)$$

Algebraic manipulation of this expression gives

$$V(s) = \frac{(s/C)I_g(s)}{s^2 + (s/RC) + (1/LC)} + \frac{s/C[Cv(0^-) - i(0^-)]}{s^2 + (s/RC) + (1/LC)} \qquad (103.21)$$

The transfer function relating the source current to the capacitor voltage is obtained directly from Eq. (103.21), with $v(0^-) = 0$ and $i(0^-) = 0$:

$$H(s) = \frac{s/C}{s^2 + (s/RC) + (1/LC)} \qquad (103.22)$$

Because $V(s)$ represents the response of a second-order circuit, the form of its partial fraction expansion depends on the natural frequencies of the circuit. These are obtained by solving for the roots of the characteristic equation:

$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0 \qquad (103.23)$$

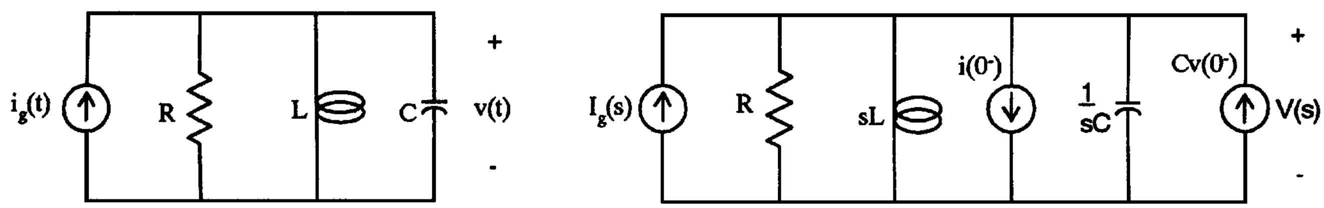
The roots are

$$s_1 = \frac{-1}{2RC} + \sqrt{\left[\frac{1}{2RC}\right]^2 - \frac{1}{LC}} \qquad (103.24)$$

$$s_2 = \frac{-1}{2RC} - \sqrt{\left[\frac{1}{2RC}\right]^2 - \frac{1}{LC}} \qquad (103.25)$$

To illustrate the possibilities of the second-order response, we let $i_g(t) = u(t)$, and $I_g(s) = 1/s$.

Figure 103.5 Parallel RLC circuit.



Case 1: Overdamped Circuit

If both roots of the second-order characteristic equation are real valued the circuit is said to be *overdamped.* The physical significance of this term is that the circuit response to a step input exhibits exponential decay and does not oscillate. The form of the step response of the circuit's capacitor voltage is

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \qquad (103.26)$$

and K_1 and K_2 are chosen to satisfy the initial conditions imposed by $v(0^-)$ and $i(0^-)$.

Case 2: Critically Damped Circuit

When the two roots of a second-order characteristic equation are identical the circuit is said to be *critically damped.* The circuit in this example is critically damped when $s_1 = s_2 = -1/(2RC)$. In this case, Eq. (103.21) becomes

$$V(s) = \frac{I/C}{[s + (1/2RC)]^2} + \frac{s}{C}\frac{[Cv(0^-) - i(0^-)]}{[s + (1/2RC)]^2} \qquad (103.27)$$

The partial fraction expansion of this expression is

$$V(s) = \frac{1/C}{[s + (1/2RC)]^2} + \frac{1}{C}\frac{[Cv(0^-) - i(0^-)]}{s + (1/2RC)} - \frac{1}{2RC^2}\frac{[Cv(0^-) - i(0^-)]}{[s + (1/2RC)]^2} \qquad (103.28)$$

Taking the inverse Laplace transform of $V(s)$ gives

$$v(t) = \frac{1}{C} t e^{-t/(2RC)} + \frac{1}{C}[Cv(0^-) - i(0^-)]e^{-t/(2RC)} - \frac{1}{2RC^2}[Cv(0^-) - i(0^-)]te^{-t/(2RC)} \qquad (103.29)$$

The behavior of the circuit in this case is called *critically damped* because a reduction in the amount of damping in the circuit would cause the circuit response to oscillate.

Case 3: Underdamped Circuit

The component values in this case are such that the roots of the characteristic equation are a complex conjugate pair of numbers. This leads to a response that is oscillatory, having a damped frequency of oscillation, ω_d, given by

$$\omega_d = \sqrt{\frac{1}{LC} + \left[\frac{1}{2RC}\right]^2} \qquad (103.30)$$

and a damping factor, α, given by

$$\alpha = \frac{1}{2RC} \qquad (103.31)$$

The form of the response of the capacitor voltage to a unit step input current source is

$$v(t) = 2|K|e^{-\alpha t}\sin(\omega_d t + \varphi) \qquad (103.32)$$

where the parameters *K* (a nonnegative constant) and φ are chosen to satisfy initial conditions.

A comparison of overdamped, critically damped, and underdamped responses of the capacitor voltage is given in Fig. 103.6 for a unit step input, with the circuit initially relaxed. Additional insight into the behavior of RL, RC, and RLC circuits can be obtained by examining the response of the current in the series RL, RC, and RLC circuits for corresponding component values. For example, Fig. 103.7 shows the waveforms of the step response of $i(t)$ for the following component values: $R = 4$ - , $L = 1$ H , and $C = 1$ F . The RL circuit has a time constant of 0.25 s, the RC circuit has a time constant of 2.00 s, and the overdamped RLC circuit has time constants of 0.29 s and 1.69 s. The inductor in the RL circuit blocks the initial flow of current, the capacitor in the RC circuit blocks the steady state flow of current, and the RLC circuit exhibits a combination of both behaviors. Note that the time constants of the RLC circuit are bounded by those of the RL and RC circuits.

Figure 103.6 Overdamped, critically damped, and underdamped responses of a parallel RLC circuit with a step input.

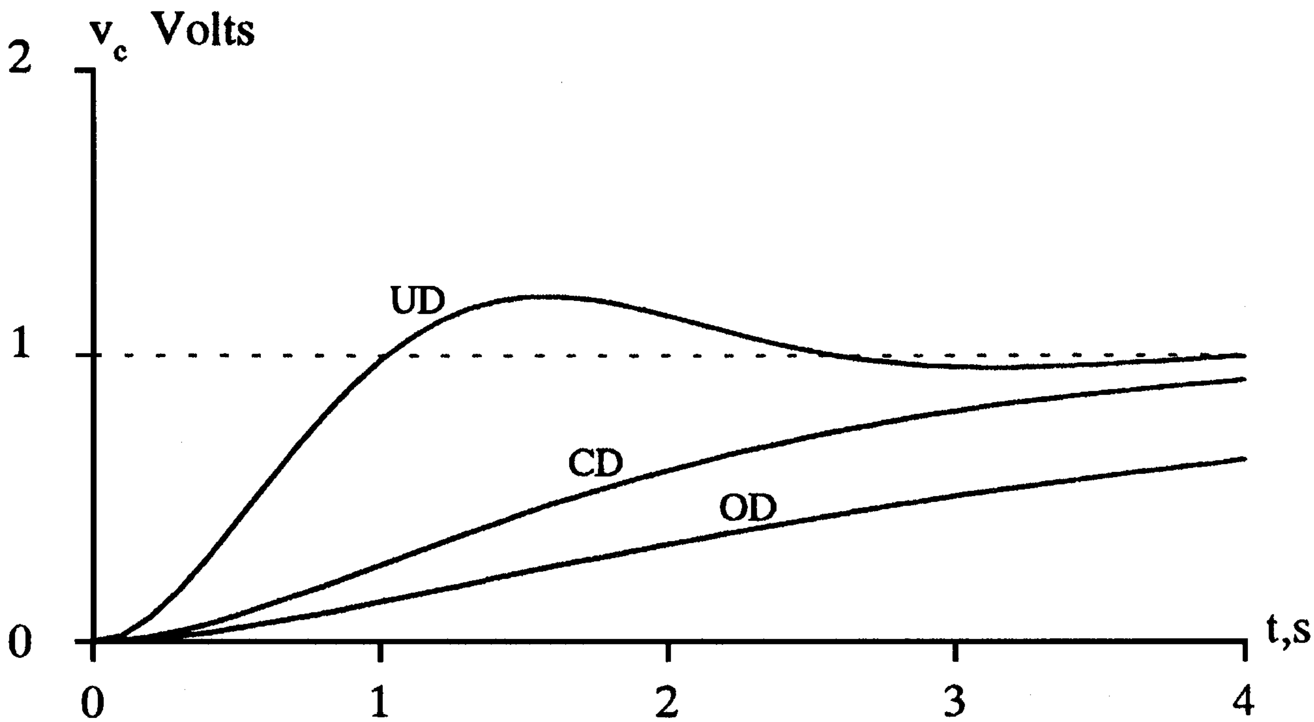
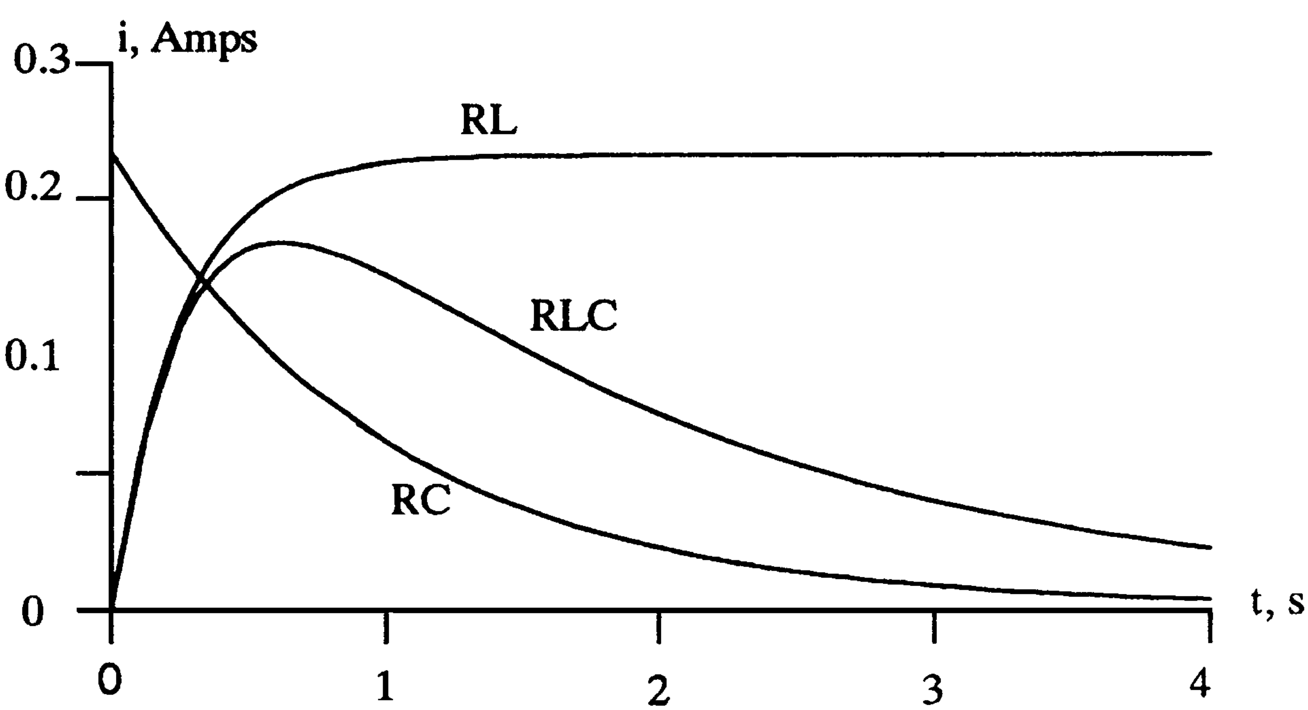


Figure 103.7 A comparison of current in series RL, RC, and RLC circuits.



RLC Circuit – Frequency Response

A circuit's transfer function defines a relationship between the frequency-domain (spectral) characteristics of any input signal and the frequency-domain characteristics of its corresponding output signal. In many engineering applications circuits are used to shape the spectral characteristics of a signal. For example, in a communications application the frequency response could be chosen to eliminate noise from a signal. The frequency response of a circuit consists of the graphs of $|H(j\omega)|$ and $\theta(j\omega)$, the magnitude response, and the phase response, respectively, of the transfer function of a given circuit voltage or current. The term $|H(j\omega)|$ determines the ratio of the amplitude of the *sinusoidal steady state response* of the circuit to a sinusoidal input, and $\theta(j\omega)$ determines the phase shift (manifest as a time-axis translation) between the input and the output waveforms. Values of $|H(j\omega)|$ and $\theta(j\omega)$ are obtained by taking the magnitude and angle, respectively, of the complex value $H(j\omega)$.

The magnitude and phase responses play an important role in filter theory, where for distortionless transmission (i.e., the filter output waveform is a scaled and delayed copy of the input waveform) it is necessary that $|H(j\omega)| = K$, a constant (flat response), and $\theta(j\omega)$ be linear over the pass band of a signal that is to be passed through a filter.

In the parallel RC circuit of Fig. 103.3, the transfer function relating the capacitor voltage to the current source has

$$H(j\omega) = \frac{R}{1 + j\omega RC} \qquad (103.33)$$

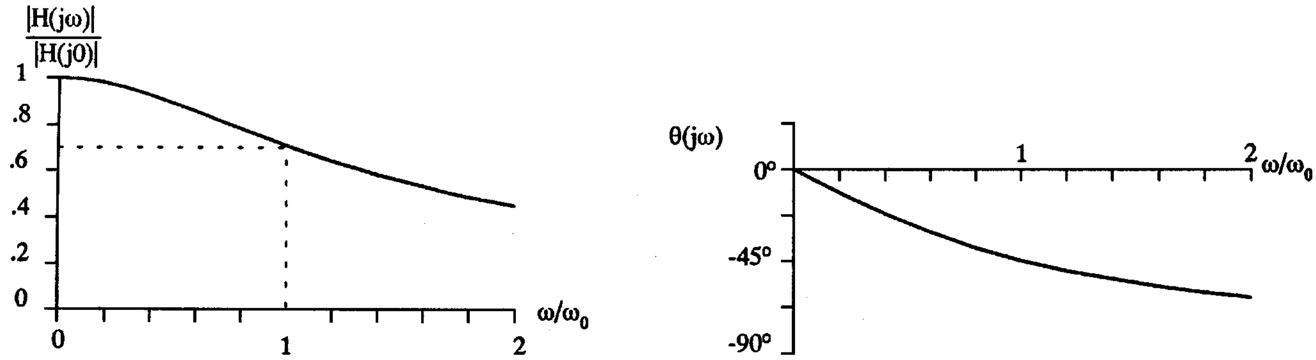
The arithmetic of complex numbers gives the following

$$|H(j\omega)| = \frac{R}{\sqrt{1 + \omega^2 R^2 C^2}} \qquad (103.34)$$

$$\theta(j\omega) = \angle H(j\omega) = -\tan^{-1}(\omega RC) \qquad (103.35)$$

The graphs of $|H(j\omega)|$ and $\theta(j\omega)$ are shown in Fig. 103.8. The capacitor voltage in the parallel RC circuit has a "low pass" filter response, meaning that it will pass low frequency sinusoidal signals without significant attenuation provided that $\omega < \omega_o [\omega_o = 1/(RC)]$, the cutoff frequency of the filter. The approximate linearity of the phase response within the passband is also evident.

Figure 103.8 Magnitude and phase response of a parallel RC circuit.



A filter's component values determine its cutoff frequency. An important design problem is to determine the values of the components so that a specified cutoff frequency is realized by the circuit. Here, increasing the size of the capacitor lowers the cutoff frequency, or, alternatively, reduces the bandwidth of the filter. Other typical filter responses that can be formed by RL, RC, and RLC circuits (with and without op amps) are high-pass, band-pass, and band-stop filters.

Defining Terms

Characteristic equation: The equation obtained by setting the denominator polynomial of a transfer function equal to zero. The equation defines the natural frequencies of a circuit.

Generalized impedance: An *s*-domain transfer function in which the input signal is a circuit's current and the output signal is a voltage in the circuit.

Natural frequency: A root of the characteristic polynomial. A natural frequency corresponds to a mode of exponential time-domain behavior.

Natural solution: The solution to the unforced differential equation.

Particular solution: The solution to the differential equation for a particular forcing function.

Steady state response: The response of a circuit after sufficient time has elapsed to allow the transient response to become insignificant.

Transient response: The response of a circuit prior to its entering the steady state.

Transfer function: An *s*-domain function that determines the relationship between an exponential forcing function and the particular solution of a circuit's differential equation model. It also describes a relationship between the *s*-domain spectral description of a circuit's input signal and the spectral description of its output signal.

References

Ciletti, M. D., 1988. *Introduction to Circuit Analysis and Design.* Holt, Rinehart and Winston, New York.

Ziemer, R. E., Tranter, W. H., and Fannin, D. R. 1993. *Signals and Systems: Continuous and Discrete.* Macmillan, New York.

Further Information

For further information on the basic concepts of RL, RC, and RLC circuits, see *Circuits, Devices,* *and Systems* by R. J. Smith and R. C. Dorf. For a treatment of convolution methods, Fourier transforms, and Laplace transforms, see *Signals and Systems: Continuous and Discrete* by R. E. Ziemer *et al.* For a treatment of RL, RC, and RLC circuits with op amps, see *Introduction to Circuit Analysis and Design* by Ciletti.