3.3.3 Normal Distribution

The normal distribution has several advantages over the other distributions.

a. The normal distribution and distributions associated with it are very tractable and analytically.

b. The normal distribution has the familiar bell shape, whose symmetry makes it an appealing choice for many popular models.

c. There is the Central Limit Theorem, which shows that, under mild conditions, the normal distribution can be used to approximate a large variety of distributions in large samples.

The normal distribution has two parameters, usually denoted by μ and σ^2, which are its mean and variance. The pdf is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty.$$

If $X \sim N(\mu, \sigma^2)$, then the random variable $Z = (X - \mu)/\sigma$ has a $N(0, 1)$ distribution, also known as the standard normal.

If $Z \sim N(0, 1)$,

$$\mathrm{E}Z = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz = 0,$$

and so, if $X \sim N(\mu, \sigma^2)$,

$$\mathrm{E}X = \mathrm{E}(\mu + \sigma Z) = \mu + \sigma \mathrm{E}Z = \mu.$$

Similarly, we have that $\mathrm{Var}Z = 1$ and $\mathrm{Var}X = \sigma^2$.

To show

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1.$$

We only need to show

$$\int_{0}^{\infty} e^{-z^2/2} dz = \sqrt{\frac{\pi}{2}}.$$

Since

$$\Big(\int_0^\infty e^{-z^2/2}dz\Big)^2 = \Big(\int_0^\infty e^{-t^2/2}dt\Big)\Big(\int_0^\infty e^{-u^2/2}du\Big)$$
$$= \int_0^\infty \int_0^\infty e^{-(t^2+u^2)/2}dtdu.$$

Now we convert to polar coordinates. Define

$$t = r\cos\theta, \quad u = r\sin\theta.$$

Then $t^2 + u^2 = r^2$ and $dtdu = rd\theta dr$ and the limits of integration become $0 < r < \infty$, $0 < \theta < \pi/2$. We now have

$$\int_0^\infty \int_0^\infty e^{-(t^2+u^2)/2}dtdu = \int_0^\infty \int_0^\infty re^{-r^2/2}d\theta dr$$
$$= \frac{\pi}{2}\int_0^\infty re^{-r^2/2}dr = \frac{\pi}{2}.$$

The probability content within 1, 2 or 3 standard deviations of the mean is

$$P(|X-\mu| \le \sigma) = P(|Z| \le 1) = .6826,$$
$$P(|X-\mu| \le 2\sigma) = P(|Z| \le 2) = .9544,$$
$$P(|X-\mu| \le 3\sigma) = P(|Z| \le 3) = .9974,$$

where $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0,1)$.

Among the many uses of the normal distribution, an important one is its use as an approximation to other distributions. For example, if $X \sim$ binomial(n,p), then $EX = np$ and $\text{Var}X = np(1-p)$, and under suitable conditions, the distribution of X can be approximated by that of a normal random variable with mean $\mu = np$ and variance $\sigma^2 = np(1-p)$. The suitable conditions are that n should be large and p should not be extreme (near 0 or 1). We want n large so that there are enough values of X to make an approximation by a continuous distribution reasonable, and p should be "in the middle" so the binomial is nearly symmetric, as is the normal. A conservative rule to follow is that the approximation will be good if $\min(np, n(1-p)) \ge 5$.