1.2.3 Counting and Equally Likely Outcomes

Methods of counting are often used in order to construct probability assignments on finite sample spaces, although they can be used to answer other questions also. The following theorem is sometimes known as the Fundamental Theorem of Counting.

- Product Rule for Ordered Pairs

  If a job consists of k separate tasks, the i^{th} of which can be done in n_i ways, $i = 1, \ldots, k$, then the entire job can be done in $n_1 \times n_2 \times \cdots \times n_k$ ways.

  - Example 3.1:

    One wish to have a trip from Sanfrancisco to NY via Dallas. There are four possible airlines from Sanfran. to Dallas: "American Air", "American West", "United Air", and "Intercontinental". And there are two possible airlines from Dallas to NY: "American Air" and "United Air". Then, how many different ways she can choose?

  - Tree Diagram:

  - Example 3.2:

    Suppose you want to repair your house and need one plumber and one electrician. There are 12 plumbers and 9 electricians in the town. How many ways you can choose a plumber and an electrician?

  - Example 3.3: Plate Number Example

- Permutation Choose k objects from a set of n objects **with ORDER** is called permutation. The number of ways choosing k objects from n objects **with ORDER** is

  $$P_{k,n} = n(n-1)(n-2)\cdots(n-k+1).$$

  - Tree Diagram:

  - Example 3.4:

    There are 10 graders for elementary engineering class, say G_1, G_2, $\cdots$, G_{10}, and 4 problem in the final exam. How many ways can graders be assigned to the final exam? (no one grade more than 1 problem)

- $m! = m \cdot (m-1) \cdot (m-2) \cdots 2 \cdot 1.$

$$P_{k,n} = \frac{n!}{(n-k)!}.$$

- Combination

  Choose k objects from a set of n objects **without ORDER** is called combination. The number of ways choosing k objects from n objects **without ORDER** is

$$\binom{n}{k} = C_{k,n} = \frac{n!}{k!(n-k)!}$$

  - Example 3.6: choose 5 cards from a deck (52 cards)

$$\binom{52}{5} = \frac{52!}{5! \cdot 47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

  - Example 3.7

    A box has 2 white balls and 3 black balls. We choose 2 balls from the box. What is the probability for one is white and one is black.

  - Example 3.8

    A box has 3 white balls and 3 black balls. We choose 3 balls from the box. What is the probability for 2 are white and 1 is black.

  - Example 3.9

    A box has four 40-W bulbs, five 60-W bulbs, and six 75-W bulbs. Choose 3 bulbs. What is the probability that one bulb of each type is selected?

  - Problem 3.10

    20 workers on day shift, 15 workers on swing shift, and 10 workers are on the graveyard shift. six of these workers are chosen. What is the probability that all selected 6 workers are from the same shift? What is the probability that at least two different shifts will be represented among the selected workers?

- Example from the Textbook For a number of years the New York state lottery operated according to the following scheme. From the numbers 1,2, ..., 44, a person may pick any six for her ticket. The winning number is then decided by randomly selecting six numbers from the forty-four. So the first number can be chosen in 44 ways, and the second number in 43 ways, making a total of $44 \times 43 = 1892$ ways of choosing the first two numbers. However, if a person is allowed to choose the same number twice, then the first two numbers can be chosen in $44 \times 44 = 1936$ ways.

The above example makes a distinction between counting with replacement and counting without replacement. The second crucial element in counting is whether or not the ordering of the tasks is important. Taking all of these considerations into account, we can construct a 2×2 table of possibilities.

	Without replacement	With replacement
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\binom{n}{r}$	$\binom{n+r-1}{r}$

Let us consider counting all of the possible lottery tickets under each of these four cases.

Ordered, without replacement From the Fundamental Theorem of Counting, there are

$$44 \times 43 \times 42 \times 41 \times 40 \times 39 = \frac{44!}{38!} = 5,082,517,440$$

possible tickets.

Ordered, with replacement Since each number can now be selected in 44 ways, there are

$$44 \times 44 \times 44 \times 44 \times 44 \times 44 = 44^6 = 7,256,313,856$$

possible tickets.

Unordered, without replacement From the Fundamental Theorem, six numbers can be arranged in 6! ways, so the total number of unordered tickets is

$$\frac{44 \times 43 \times 42 \times 41 \times 40 \times 39}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{44!}{6!38!} = 7,059,052.$$

Unordered, with replacement In this case, the total number of unordered tickets is

$$\frac{44 \times 45 \times 46 \times 47 \times 48 \times 49}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{49!}{6!43!} = 13,983,816.$$