PART I

DISCRETE STRUCTURES AND COMPUTING

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The Apportionment Problem

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Prerequisites: The prerequisites for this chapter are sorting, big-O notation, mathematical induction, and counting. See Sections 3.1, 3.2, 4.1, and 5.1 of *Discrete Mathematics and Its Applications*.

Introduction

How should I divide a bag of jelly beans among my six nephews? How should Uncle Sam divide his military bases among his fifty "nephews"? How should the House of Representatives assign its 435 seats to the fifty states which make up this union? These are different *apportionment problems*, which are made difficult by the assumption that the apportionment should be "fair".

The apportionment problem is a problem which has no one correct answer. Instead of being given an answer to a problem, the reader will learn how to ask questions which can be answered, and make use of the answers. This process includes deciding upon objectives, building algorithms to obtain those objectives, determining whether the algorithms will work within reasonable time constraints, and determining when two algorithms achieve the same final output. Apportionment is an especially timely topic following the decennial census (1990, 2000, etc.) when the U. S. House of Representatives is reapportioned.

Historical Background

The Constitution of the United States reads

> "Representatives . . . shall be apportioned among the several States . . ., according to their respective Numbers, which shall be determined . . .".

It is not specified how this apportionment is to be achieved. The only requirements for the apportionment are that each state shall have at least one representative, but not more than one per 30,000 residents. The total number of seats in the House of Representatives, which is determined by Congress, is at present 435. Prior to the reapportionment based on the 2000 census, several states had only one representative, while California had the largest delegation with 52 representatives. (The United States Constitution specifies that each state have two senators, hence there is no problem determining how many senators to assign to each state.)

The Problem

The basic apportionment problem is how to "fairly" divide discrete objects among competing demands. There are many examples other than those given in the introduction including the number of teachers each school in a district is assigned, or the number of senators each college in a university has on the university senate. We shall focus on dividing the seats in the U. S. House of Representatives among the states. The crux of the problem is that although "one man, one vote" (proportional representation) will usually require that states have a fractional number of representatives, an integral number of representatives must be assigned to each state. The third Congress (formed in 1793) was the first time a census (the first census of the United States, taken in 1790) was used to apportion seats in the House; we shall consider the composition of the first Congress, which is specified in the Constitution, contrasted to various "fair" apportionments based on the 1790 census, in the sequel.

Example 1 Let n be the number of states entitled to representatives. For example, $n = 13$ when the United States was formed, $n = 50$ today. We denote the population of the ith state by p_i and let

$$P = \sum p_i$$

the total population of the United States. In 1790 the population of Connecticut was 236,840.4 (this is not an integer because slaves were counted as 0.6 each), and the total population was 3,596,687. Therefore, Connecticut should have received

$$\frac{236{,}840.4}{3{,}596{,}687} = 6.58\%$$

of the representatives. The total size of the House of Representatives, which will be denoted by H, was 65 in 1789. Therefore Connecticut should have received

$$0.0658 \cdot 65 = 4.28$$

representatives. The **quota** for the ith state will be denoted by

$$q_i = \frac{p_i}{P} \cdot H,$$

so that $\sum q_i = H$. We will denote the actual number of representatives allotted to the ith state (which must be an integer) by a_i, subject to the constraint $\sum a_i = H$. The Constitution specifies the number of representatives each state was entitled to before the first (1790) census; the Constitution assigned Connecticut five representatives prior to the first census. (The complete apportionment specified by the Constitution for use prior to the first census is given in Table 1 in the column labeled "C".) $\square$

If an apportionment is not externally mandated (e.g., specified in the Constitution), we must construct one in a "fair" manner. In order to illustrate the construction of apportionments, it is useful to assume fewer than 13 states with population sizes which do not distract from the concepts being presented.

Example 2 Consider a mythical country with four states A, B, C, and D, with populations 1000, 2000, 3000, and 4000 respectively. The total number of seats for this country is 13 ($H = 13$). Find the quota for each state and choose integers $\{a_i\}$ which provide a "fair" apportionment.

Solution: A "fair" apportionment should have the a_i as close to the q_i as possible; i.e., the integer number of seats assigned to each state should be as close as possible to its quota, which in general will not be an integer. This is easily obtained by rounding off the quota for each state. For our fictitious country the quotas are

$$\frac{1000}{10000} \cdot 13 = 1.3,$$

$$\frac{2000}{10000} \cdot 13 = 2.6,$$

$$\frac{3000}{10000} \cdot 13 = 3.9,$$

$$\frac{4000}{10000} \cdot 13 = 5.2.$$

These round off to the integers 1, 3, 4, and 5 for the actual number assigned, a_i. However this procedure is naïve in that it may not yield the specified house size. In particular, Table 1 shows that this method would have provided 66 representatives for the 1790 census. □

Therefore, another criterion of closeness must be employed. One method is to construct a global index of closeness and then find the apportionment which optimizes it. The term **global index** means that all the deviations of the a_i from the q_i are combined in some manner to form a single number which serves as a measure of fairness; this contrasts the pairwise criteria of fairness discussed below. Examples of global indices include

$$\sum |a_i - q_i|$$

and

$$\sum |a_i - q_i|^2.$$

Minimizing either of these two indices provides a sense of fair apportionment.

In Example 2, where $H = 13$ and $\mathbf{p} = (1000, 2000, 3000, 4000)$, these indices are equal to

$$0.3 + 0.4 + 0.1 + 0.2 = 1$$

and

$$0.09 + 0.16 + 0.01 + 0.04 = 0.3,$$

respectively, for our fictitious country with the apportionment 1, 3, 4, 5. It can easily be shown that for this case these values provide the minima for the respective indices. However, especially for more complicated indices, we may need to evaluate the index for every possible apportionment of the seats to the states to find the apportionment which will minimize the index, and hence provide the fairest apportionment. Even if we could decide which index best embodies "fairness", it is not feasible to check all possible apportionments because there are too many apportionments to check.

Example 3 Find the number of possible apportionments that need to be checked for the country of Example 2.

Solution: The formula for combinations with repetition (Section 5.5 of *Discrete Mathematics and Its Applications*) can be modified to find the number of possible apportionments which must be checked. We first assign one representative to each state (as mandated by the Constitution) and then divide the remaining seats among the states (sample with replacement from among the states until the House is filled). An alternative motivation for the formula lines up the H seats in the House so that there are $H - 1$ spaces between them and then chooses $n-1$ of the spaces to demarcate the number of seats each state receives. For our fictitious country with four states and thirteen seats there are $12!/(3!9!) = 220$ possible apportionments. □

Example 4 For 13 states and a House with 65 seats, the same reasoning yields

$$\frac{64!}{12!\,52!} \approx 3.3 \times 10^{12}$$

apportionments to check, which is about 10^5 cases for every man, woman, and child in the United States in 1790. Although we now have computers to do calculations, the current size of the House and current number of states leaves checking all possible apportionments unfeasible because the number of cases far exceeds the power of our computers. (Estimating the current number of possible apportionments is left as Exercise 12.) □

Algorithms for Apportionment

Construction of indices of fairness is easy, but they are not useful unless there is a means to find the apportionment which optimizes them. To apportion the House fairly, we need to have an algorithm to obtain that end. Several algorithms for apportioning the House are presented below. Reconciliation of these algorithms with the fairness of the resultant apportionments is addressed later.

Largest fractions

The problem with rounding off the quota to determine the number of assigned seats, i.e., setting

$$a_i = \lfloor q_i + .5 \rfloor,$$

is that too few or too many seats may be assigned, as was noted for the quota column in Table 1. An easy way to correct this problem is to assign each state the integer part of its quota, $\lfloor q_i \rfloor$, and then assign the remaining seats to the states with the largest fractional parts until the House is full. If rounding off produces a House of the proper size, this method provides the same apportionment. This method is known as the method of **largest fractions**, and also known as the **Hare quota** method. Alexander Hamilton was a proponent of this method, but its use was vetoed by George Washington. (It was subsequently used from 1850–1910, at which time it was referred to as the **Vinton method**, named after Congressman S. F. Vinton*.) This method is easy to implement: we merely need to calculate the quota for each state and then rank order the fractional parts. The complexity of this algorithm is governed by the sort which rank orders the fractional parts, since calculating all the quotas entails only $O(n)$ operations, but the complexity of a sort of n objects is $O(n \log n)$. (See Section 10.2 of *Discrete Mathematics and Its Applications*.) Although this algorithm is concise, it is not useful until the resultant apportionment has been shown to be "fair".

Example 5 Find the largest fractions apportionment for the fictitious country from Example 2.

Solution: The quotas are

$$1.3,\ 2.6,\ 3.9,\ 5.2.$$

Rounding down guarantees the states 1, 2, 3, and 5 representatives, respectively, which leaves 2 more to be assigned. The largest fractional part is 0.9 which belongs to state C and the second largest fractional part is 0.6 which belongs to state B. Hence the largest fractions apportionment is 1, 3, 4, 5. □

The apportionment for a House of 65 seats based on the 1790 census using the method of largest fractions is given in Table 1 in the column labeled "LF".

λ–method

The λ–method is a generalization of rounding off the quota which provides that the requisite number of seats will be assigned. There are several alternative versions (which often produce different apportionments). All of the versions entail finding a λ to modify the quotas so that the proper number of seats will be assigned. One version, known as the method of **major fractions**, or the method of the **arithmetic mean**, was advocated by Daniel Webster; it was used to apportion the House of representatives from 1911–1940 when it was advocated by Walter F. Willcox of Cornell University. To assign H seats, we need to find a λ such that

$$\sum \left\lfloor \frac{q_i}{\lambda} + .5 \right\rfloor = H$$

and set

$$a_i = \left\lfloor \frac{q_i}{\lambda} + .5 \right\rfloor .$$

* S. F. Vinton (1792–1862) was a congressman from Ohio who also served as president of a railroad.

State	Population	Quota	C	LF	GD	MF	EP	SD
NH	141,821.8	2.56	3	2	2	3	3	3
MA	475,327.0	8.59	8	9	9	8	9	8
RI	68,445.8	1.24	1	1	1	1	1	2
CT	236,840.4	4.28	5	4	4	4	4	4
NY	331,590.4	5.99	6	6	6	6	6	6
NJ	179,569.8	3.25	4	3	3	3	3	3
PA	432,878.2	7.82	8	8	8	8	8	8
DE	55,541.2	1.00	1	1	1	1	1	1
MD	278,513.6	5.03	6	5	5	5	5	5
VA	699,264.2	12.63	10	13	13	13	12	12
NC	387,846.4	7.01	5	7	7	7	7	7
SC	206,235.4	3.73	5	4	4	4	4	4
GA	102,812.8	1.86	3	2	2	2	2	2

Table 1. Apportionment of the first House of Representatives, based on the 1790 census. Slaves were counted as 0.6. The apportionment specified in the Constitution (C) is followed by those given by the methods of largest fractions (LF), greatest divisors (GF), major fractions (MF), equal proportions (EP), and smallest divisors (SD).

Example 6 The value λ=1 works for our fictitious country of Example 2, since rounding off assigns the proper number of seats. However, rounding off provides 66 seats for the 1790 census with a House size of 65. By trial and error, we discover that any λ in the range 1.0107–1.0109 will provide a House with 65 seats as given in the column labeled "MF" in Table 1. □

A variant known as the method of **greatest divisors**, or the method of **rejected fractions**, was advocated by Thomas Jefferson and used from 1791–1850. This method requires that we find a λ such that

$$\sum \left\lfloor \frac{q_i}{\lambda} \right\rfloor = H$$

and set

$$a_i = \left\lfloor \frac{q_i}{\lambda} \right\rfloor.$$

The apportionment based on the 1790 census using this method is given in the column labeled "GD" in Table 1. Another variant, for which John Quincy Adams was a proponent, is the method of **smallest divisors**. For this method we find a λ such that

$$\sum \left\lceil \frac{q_i}{\lambda} \right\rceil = H$$

and set

$$a_i = \left\lceil \frac{q_i}{\lambda} \right\rceil.$$

The apportionment based on the 1790 census using the method of smallest divisors is given in the column labeled "SD" in Table 1. The method of smallest divisors was never used to apportion the House of Representatives.

The time required to apportion the House by a λ–method depends on the particular algorithm used. A procedure analogous to bisecting an interval to find a root of an equation can be employed. Because there is an interval of λs which provide the proper House size (e.g., [1.0107, 1.0109] in Example 6), a value of λ for which the summation equals H (i.e., a $\lambda \in [1.0107, 1.0109]$) will be obtained after enough successive approximations. (The exact root of an equation is seldom found by the bisection method because there is often only an irrational root which is never reached by bisection.) For each λ which is tried, n summands must be calculated, which require a fixed number of operations each; then the summands must be added. Hence $O(n)$ calculations must be performed for each λ. The total complexity depends on how many λs it may be necessary to try. Values of λ which work may range from the number of states (n) to its reciprocal ($1/n$), but in general choosing 1 and then a number near 1 will serve as good initial choices for λ with the bisection method. (Of course, the second choice should be greater than 1 if $\lambda = 1$ assigned too many seats, and conversely.)

Huntington Sequential Method

An alternative approach for achieving the optimal apportionment is to start with no seats assigned (or with one seat allocated to each state), and then sequentially assign each seat to the "most deserving" state. E. V. Huntington* reformulated previously employed and proposed methods in this framework, and also proposed new methods of apportionment. These apportionment methods differ by the concept of "most deserving" they employ. Algorithm 1 incorporates the criterion of "most deserving" which Huntington favored. It is known as the method of **equal proportions** or the method of the **harmonic mean** and has been used since 1941 to apportion the U. S. House of Representatives.

* E. V. Huntington (1874–1952) was a Professor of Mechanics at Harvard University, and his many professional positions included presidency of the Mathematical Association of America in 1918. *Survey of Methods of Apportionment in Congress* (Senate document 304; 1940) is one of his lesser-known publications.

ALGORITHM 1 The method of equal proportions.

```
procedure Huntington(H: size of House; n: number of
   states; p_i: population of state i)
for i := 1 to n
   a_i := 1
   {a_i is the number of seats assigned; start at 0 for some
   variations}
s := n
{s is sum of assigned representatives; use 0 if 0 used for a_i}
while s < H
begin
   maxr := 0
   {maxr is the largest value of any r_i}
   maxindex := 1
   {maxindex is the subscript of an r_i that equals maxr}
   for i := 1 to n
   begin
      r_i := p_i/sqrt(a_i(a_i + 1))
      {r_i measures how deserving state i is}
      if r_i > maxr then
      begin
         maxr := r_i
         maxindex := i
      end
   end
   a_maxindex := a_maxindex + 1
   {next seat is assigned to most deserving state}
   s := s + 1
end {a_i is the number of representatives assigned to state i}
```

The largest value of r_i indicates that the smallest number of representatives relative to its population has been assigned to that state, hence that state merits the next seat. It is not obvious whether the next seat should be given to the state which is most underrepresented now, or would be most underrepresented if given another seat; taking the geometric mean of a_i and $a_i + 1$ as illustrated in Algorithm 1 is a compromise of these two philosophies in that it provides an average of the merit under each philosophy.

Changing the definition of r_i provides different methods of apportionment. Employing the **arithmetic mean** of a_i and a_i+1 in the denominator instead of the geometric mean produces the **major fractions** apportionment; employing the **harmonic mean** produces an apportionment which has been advocated, but has not been used, for apportioning the House of Representatives. If a_i alone is in the denominator, the **smallest divisors** apportionment results, and if $a_i + 1$ alone is in the denominator, the **greatest divisors** apportionment results.

The number of operations required to calculate all the r_i is $O(n)$, since each requires only a few algebraic operations. (Specifically, perhaps adding 1 to the a_i of record, perhaps averaging that sum with a_i in some manner, and then dividing p_i by the resultant.) To find the maximum of the r_i, $O(n)$ comparisons are required. (See Section 3.1 of *Discrete Mathematics and Its Applications* for finding a maximum and for a bubble sort which will find the maximum with a single pass.) Since this must be done for each of the H seats in the House of Representatives, a total of $O(nH)$ operations are required for this algorithm. We cannot compare the efficiency of the λ and sequential algorithms because we do not have a bound on how many λs we will need to try. In the case of equal proportions there is no λ analog. (See Table 2.)

Reconciliation

We first presented fairness criteria which we could not achieve, and then algorithms for which we had no way to assess the fairness of the resultant apportionments. It is indeed possible to reconcile the algorithms which have been discussed to indices of fairness.

Largest fractions

The method of largest fractions minimizes $\sum |a_i - q_i|$. This is readily verified by first noting that if $\sum |a_i - q_i|$ is minimized, each a_i equals either $\lfloor q_i \rfloor$ or $\lceil q_i \rceil$. To see this, suppose some a_i is greater than $\lceil q_i \rceil$. Then the sum can be reduced by subtracting 1 from a_i and adding 1 to an a_j where $a_j < q_j$. (Such a state must exist since $\sum a_i = \sum q_i = H$.) If some a_i is less than $\lfloor q_i \rfloor$, the sum can be reduced by adding 1 to a_i and taking 1 from a state with $a_j > q_j$. (Such a state must exist since $\sum a_i = \sum q_i = H$.) If all the assigned numbers a_i, $i = 1, 2, \ldots, n$, result from either rounding up or rounding down the quota, then the summation is clearly minimized by rounding up those with the largest fractional parts. It also follows that the resultant apportionment minimizes the maximum value of $|a_i - q_i|$ taken over all states i. This follows since the maximum with respect to i of $|a_i - q_i|$ is the maximum of the $|a_i - q_i|$ where $a_i > q_i$ (where the quotas were rounded up) or the maximum of the $|a_i - q_i|$ where $a_i < q_i$ (where the quotas were rounded down). Any reassignment of states to the categories "round up" and "round down" will result both in

Method	When used	Proponent	Global minimization	Pairwise minimization	λ-method $H =$	r_i
LF	1850–1910	Hamilton	$\sum \lvert a_i - q_i \rvert$			
GD	1791–1850	Jefferson	$\max_i \frac{a_i}{p_i}$		$\sum \lfloor \frac{q_i}{\lambda} \rfloor$	$\frac{p_i}{a_i+1}$
MF	1911–1940	Webster	$\sum \frac{(a_i-q_i)^2}{q_i}$	$\left\lvert \frac{a_i}{p_i} - \frac{a_j}{p_j} \right\rvert$	$\sum \lfloor \frac{q_i}{\lambda} + .5 \rfloor$	$\frac{p_i}{.5(a_i+(a_i+1))}$
EP	1941–present	Huntington	$\sum \frac{(a_i-q_i)^2}{a_i}$	$\left\lvert \log \frac{a_i/p_i}{a_j/p_j} \right\rvert$		$\frac{p_i}{\sqrt{a_i(a_i+1)}}$
SD	never	J.Q.Adams	$\max_i \frac{p_i}{a_i}$		$\sum \lceil \frac{q_i}{\lambda} \rceil$	$\frac{p_i}{a_i}$

Table 2. The apportionment methods employed are given with the sense in which they are fair and alternative algorithms to achieve them.

rounding up a smaller fractional part and in rounding down a larger fractional part than when the largest fractional parts are rounded up and the smallest fractional parts are rounded down. (Because the House size H is fixed, so is the number of fractional parts which are rounded up.) This correspondence between the algorithms and optimality criteria for largest fractions is given in Table 2.

Greatest Divisors

For the method of greatest divisors (or rejected fractions), we will first show that the λ–method with rounding down and the Huntington sequential method with $a_i + 1$ in the denominator produce the same apportionment. Then we will show that this apportionment maximizes the minimum size of congressional districts, i.e., maximizes the minimum over i of p_i/a_i. (This can be interpreted alternatively as minimizing the maximum power of a voter, i.e., minimizing the maximum over i of a_i/p_i.) For demonstrating the equivalence of the λ and sequential algorithms, we shall reformulate the λ–method as finding μ such that

$$\sum \left\lfloor \frac{p_i}{\mu} \right\rfloor = H \qquad (\mu = \lambda \cdot P/H).$$

This converts the λ–method into a sequential method as μ is decreased.

Theorem 1 The lambda and sequential algorithms for greatest divisors provide the same apportionment.

Proof: We will prove this theorem using induction on H, the number of seats in the House. The proposition to be proved for all H is $P(H)$, the lambda and sequential algorithms for greatest divisors provide the same apportionment for a House of size H. If $H = 0$, no state has any representative, and the two assignment methods give the same result. Assume that they give the same result for House size H. Then for House size $H + 1$, μ will have to be further reduced until rounding down gives a state another representative. At that value of μ, $p_i/\mu = a_i + 1$ for the state i which will get another seat and $p_j/\mu < a_j + 1$ for all other states j. Therefore $p_i/(a_i + 1) = \mu$ for the state which gets the next seat and $p_j/(a_j + 1) < \mu$ for all other states. Hence the Huntington sequential algorithm assigns the same state the next seat as the lambda method. This proves $P(H)$ implies $P(H+1)$, which, with the verity of $P(0)$ mentioned above, completes the proof by induction on H. ∎

Theorem 2 The greatest divisors apportionment maximizes the minimum district size. (This can be interpreted as assuring that the most overrepresented people are as little overrepresented as possible.)

Proof: We will prove this by induction on k, the number of seats which have been assigned. The proposition to be proven for all k is $P(k)$, the greatest divisors apportionment maximizes the minimum district size when k seats are assigned.

Before any seats are assigned, all the ratios p_i/a_i (i.e., district sizes) are infinite because the denominators are zero. This shows that $P(0)$ is true. The first seat is assigned to the state for which $p_i/(0 + 1)$ is greatest, which ratio becomes the minimum district size and is the largest possible value for the minimum district size. This proves $P(1)$. If after k of the seats have been assigned the minimum district size is as large as possible, the next state to receive a seat will have the new minimum district size. (If a state which did not receive the $(k+1)$st seat retained the minimum district size, then giving one of its seats to the state which received the $(k + 1)$st seat while only k seats were assigned would would have increased the minimum district size when k seats were assigned. This contradicts the induction hypothesis that the minimum district size was maximized for k seats assigned.) The new minimum district size will be $p_j/(a_j + 1)$ for that state, where j was chosen as the state which gave the maximum value for that ratio. This completes the proof by induction on the number of seats. ∎

The method of greatest divisors sequentially assigns seats to the states which would be least represented with an additional seat, which anticipation maximizes the minimum district size.

Equal proportions

Huntington employed pairwise concepts of fairness rather than global indices of fairness for his apportionment method. Rather than measuring the total disparity in representation among all states, he posited that an apportionment was fair if no disparity between two states could be lessened by reassigning a representative from one of them to the other. For the method of equal proportions, which he favored, the criterion is to pairwise minimize

$$\left|\log \frac{a_i/p_i}{a_j/p_j}\right|.$$

This quantity employs the ratio rather than the difference between states of representatives per person (or, alternatively, congressional district sizes) to measure disparity in representation. If the representation were equal, this ratio would equal 1, and hence its logarithm would equal 0. There is also a global index,

$$\sum \frac{(a_i - q_i)^2}{a_i},$$

which equal proportions minimizes, but no intuitive interpretation of this global index is readily available.

Optimization criteria corresponding to the other apportionment methods discussed earlier are given in Table 2. The method of major fractions minimizes

$$\left|\frac{a_i}{p_i} - \frac{a_j}{p_j}\right|$$

(the absolute disparity in representatives per person) pairwise, and minimizes the global index $\sum (a_i - q_i)^2/q_i$ (subject to the constraint $\sum a_i = H$). There are also pairwise criteria for the methods of smallest divisors and greatest divisors, but they are difficult to interpret and are omitted from Table 2.

Problems with Apportionment

The most obvious obstacle to providing the best apportionment is that different methods provide different apportionments. A survey of Table 1 reveals that there is a bias which changes from favoring the large states under greatest divisors to favoring the small states under smallest divisors. This bias can be demonstrated rigorously, but the mathematics is beyond of the scope of this chapter. Is it a mere coincidence that Thomas Jefferson (who favored greatest divisors) was from the Commonwealth of Virginia, and John Quincy Adams (who favored smallest divisors) was from the Commonwealth of Massachusetts, whose neighbors included the states of New Hampshire and Rhode Island and Providence Plantations? Since no incontrovertible argument favoring one method exists, states will always argue for a method which gives them more seats.

What has come to be called the *Alabama Paradox* was discovered in 1881 when tables prepared to show the composition of the House for various sizes showed that Alabama would have 8 representatives if the House had 299 members, but only 7 representatives if the House had 300 members. Several other states were affected in a similar manner, which caused extensive rhetoric on the Senate floor and the ultimate replacement of the method of largest fractions by major fractions in 1911. It is obvious that sequential assignment methods will not produce an Alabama paradox, but there are also problems associated with those methods.

It is certainly intuitive that a_i should equal $\lfloor q_i \rfloor$ or $\lceil q_i \rceil$. If a_i is less than the former or more than the latter, the apportionment is said to *violate lower quota* or *violate upper quota*, respectively. For example, following the census of 1930, when the method of major fractions was being used, New York received more than its upper quota. The method of greatest divisors cannot violate lower quota since the choice $\lambda = H$ gives each state its lower quota, and decreasing λ to achieve a full House cannot reduce any assignments. However, the method of greatest divisors can violate upper quota.

Example 7 Suppose $H = 20$ with quotas 2.4, 3.4, 3.4, and 10.8 for four states. The value $\lambda = 0.9$ provides the apportionment 2, 3, 3, 12 for the method of greatest divisors. This apportionment violates upper quota. □

The method of largest fractions never violates quota, but violation of quota is possible under all the sequential assignment methods in Table 2 including the method of equal proportions which is currently used.

There are other properties which an apportionment method could satisfy. For example, when the House size is fixed a state should not lose a seat when its quota, q_i, increases. However, it is quite unreasonable to expect this to hold: if it held, a given quota would always assign the same number of seats independent of the quotas of the other states. The less restrictive condition that a state whose quota increases should not lose a seat to a state whose quota decreases is more reasonable. Unfortunately, it has been shown that there is no apportionment method which never violates quota, never decreases a state's delegation when the house size increases, and never gives a seat to a state whose quota decreased from one whose quota increased. Hence, we must prioritize what properties we want our apportionment method to have. The method of equal proportions which is currently used can violate quota, but satisfies the other two properties.

Suggested Readings

1. M. Eisner, "Methods of Congressional apportionment", UMAP Module 620, COMAP, Lexington, MA, 1982.

2. E. Huntington, "A New Method of Apportionment of Representatives", *Quarterly Publication of the American Statistical Association*, 1921, pp. 859–870.

3. E. Huntington, "The Apportionment of Representatives in Congress", *Transactions of the American Mathematical Society 30*, 1928, pp. 85–110.

4. W. Lucas, "Fair Division and Apportionment", Chapter 12 in L. Steen, ed. *For All Practical Purposes*, W. H. Freeman, New York, 1988, pp. 230–249.

5. W. Lucas, "The Apportionment Problem" in S. Brams, W. Lucas, and P. Straffin, ed. *Political and Related Models*, Springer-Verlag, New York, 1983, pp. 358–396.

Exercises

1. Suppose a fictitious country with four states with populations 2000, 3000, 5000, and 8000 has a house with eight seats. Use each of the apportionment methods listed in Tables 1 and 2 to apportion the house seats to the states.

2. Consider the same country as in Exercise 1, but with nine seats in the house.

3. The Third Congress had 105 seats in the House. For apportioning the Third Congress, the population figures in Table 1 must be modified by reducing North Carolina to 353,522.2 to reflect the formation of the Southwest Territory (which eventually became Tennessee), reducing Virginia to 630,559.2 to reflect the formation of Kentucky, adding Kentucky at 68,705, and adding Vermont at 85,532.6 (there were no slaves in Vermont, but 16 reported as "free colored" were recorded as slave). Find the apportionment for the Third Congress according to each of the five methods listed in Tables 1 and 2. (This can be done with a hand calculator, but computer programs would save time.)

4. Suppose that $n \geq H$. Which of the five apportionment methods assure at least one seat for each state?

5. a) Modify both the λ and Huntington sequential methods for the greatest divisors apportionment to assure at least 3 representatives per state.
b) Show that they produce the same result.

6. Use the algorithms constructed in Exercise 5 to apportion a 15-seat house for the fictitious country of Example 2.

7. Show that the λ and sequential algorithms for the smallest divisors apportionment produce the same result.

8. Show that the λ and sequential algorithms for the smallest divisors apportionment minimize $\max_i(p_i/a_i)$.

9. a) Show that smallest divisors can violate lower quota.
b) Can it violate upper quota?

10. a) Construct an example illustrating the Alabama paradox. (Recall that this can only happen under the largest fractions apportionment.)
b) Can this happen if there are only two states?

11. How many apportionments are possible with 15 states and 105 seats if each state has at least one seat?

12. How many apportionments are possible with 50 states and 435 seats if each state has at least one seat?

13. Solve Exercise 3, but for the 1990 census with a House size of 435. (Visit http://www.census.gov/ for the 1990 census data. *Computer programs are strongly recommended for this problem.*) Which of the five methods would benefit your state the most?

14. (Calculus required) The denominators in the Huntington sequential method (labeled r_i in Table 2) involve averages of the form

$$(.5(a_i^t + (a_i + 1)^t))^{1/t}.$$

(The choices $t = 1$, 0, and -1 provide the arithmetic, geometric, and harmonic means, respectively.)
a) Show $\lim_{t\to 0}(.5(a_i^t+(a_i+1)^t))^{1/t} = \sqrt{a_i(a_i+1)}$ (the geometric mean).
b) Show that $\lim_{t\to\infty}(.5(a_i^t + (a_i + 1)^t))^{1/t} = a_i + 1$ (the maximum).
c) Show that $\lim_{t\to-\infty}(.5(a_i^t + (a_i + 1)^t))^{1/t} = a_i$ (the minimum).

Computer Projects

1. Write a computer program to apportion the House by the method of major fractions using the lambda method.
2. Write a computer program to apportion the House by the method of major fractions using the sequential (r_i) method.
3. Write a program which will check all apportionments consistent with rank order of size (no state which is smaller than another state will receive more seats, but equality of seats is allowed).