

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 4.3—Recursive Definitions and Structural Induction

 — Page references correspond to locations of Extra Examples icons in the textbook.

p.295, icon at Example 1

#1. Suppose $f(n+1) = \left\lfloor \frac{n^2 f(n) + 2}{n+1} \right\rfloor$ and $f(0) = 2$. Find $f(1)$, $f(2)$, $f(3)$, $f(4)$.

Solution:

To find $f(1)$, we use $n = 0$: $f(1) = f(0+1) = \left\lfloor \frac{0^2 f(0) + 2}{0+1} \right\rfloor = \left\lfloor \frac{0^2 \cdot 2 + 2}{0+1} \right\rfloor = \left\lfloor \frac{2}{1} \right\rfloor = 2$.

To find $f(2)$, we use $n = 1$: $f(2) = f(1+1) = \left\lfloor \frac{1^2 f(1) + 2}{1+1} \right\rfloor = \left\lfloor \frac{1^2 \cdot 2 + 2}{1+1} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor = 2$.

To find $f(3)$, we use $n = 2$: $f(3) = f(2+1) = \left\lfloor \frac{2^2 f(2) + 2}{2+1} \right\rfloor = \left\lfloor \frac{2^2 \cdot 2 + 2}{2+1} \right\rfloor = \left\lfloor \frac{10}{3} \right\rfloor = 3$.

To find $f(4)$, we use $n = 3$: $f(4) = f(3+1) = \left\lfloor \frac{3^2 f(3) + 2}{3+1} \right\rfloor = \left\lfloor \frac{3^2 \cdot 3 + 2}{3+1} \right\rfloor = \left\lfloor \frac{29}{4} \right\rfloor = 7$.

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#2. Suppose

$$f(n) = \begin{cases} f(n-2) & \text{if } n \text{ is even} \\ f(n-2) + 3 & \text{if } n \text{ is odd.} \end{cases}$$

Also suppose that $f(0) = 1$ and $f(1) = 4$. Find $f(7)$.

Solution:

Using the recurrence relation, we obtain $f(3) = 7$, $f(5) = 10$, and $f(7) = 13$.

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#3. Prove that the following proposed recursive definition of a function on the set of nonnegative integers fails to produce a well-defined function.

$$f(n) = \begin{cases} f(n-2) & \text{if } n \text{ is even} \\ 3f(n-2) & \text{if } n \text{ is odd} \end{cases}$$

with $f(0) = 4$.

Solution:

The value $f(1)$ cannot be computed because $f(1) = 3f(-1)$, but $f(-1)$ is not defined.

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#4. Prove that the following proposed recursive definition of a function on the set of nonnegative integers fails to produce a well-defined function.

$$f(n) = f(f(n-1)) + 5, f(0) = 1.$$

Solution:

When we try to compute $f(1)$, we obtain

$$f(1) = f(f(0)) + 5 = f(1) + 5,$$

which cannot happen.

p.297, icon at Example 6

#1. For the sequence of Fibonacci numbers f_0, f_1, f_2, \dots ($0, 1, 1, 2, 3, 5, 8, 13, \dots$), prove that

$$f_0 + f_2 + f_4 + f_6 + \dots + f_{2n} = f_{2n+1} - 1$$

for all $n \geq 0$.

Solution:

Let $P(n)$ be: $f_0 + f_2 + f_4 + f_6 + \dots + f_{2n} = f_{2n+1} - 1$.

BASIS STEP: $P(0)$ states that $f_0 = f_1 - 1$, which is true because $f_0 = 0$ and $f_1 - 1 = 1 - 1 = 0$.

INDUCTIVE STEP: $P(k) \rightarrow P(k+1)$: Suppose that $P(k)$ is true; i.e., $f_0 + f_2 + f_4 + f_6 + \dots + f_{2k} = f_{2k+1} - 1$. We must show that $f_0 + f_2 + f_4 + f_6 + \dots + f_{2(k+1)} = f_{2(k+1)+1} - 1$, i.e., $f_0 + f_2 + f_4 + f_6 + \dots + f_{2k+2} = f_{2k+3} - 1$:

$$\begin{aligned} f_0 + f_2 + f_4 + f_6 + \dots + f_{2k} + f_{2k+2} &= (f_0 + f_2 + f_4 + f_6 + \dots + f_{2k}) + f_{2k+2} \\ &= (f_{2k+1} - 1) + f_{2k+2} \\ &= f_{2k+1} + f_{2k+2} - 1 \\ &= f_{2k+3} - 1. \end{aligned}$$

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#2. For the sequence of Fibonacci numbers f_0, f_1, f_2, \dots ($0, 1, 1, 2, 3, 5, 8, 13, \dots$), prove for all nonnegative integers n :

$$f_0^2 + f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}.$$

Solution:

Let $P(n)$ be the proposition

$$f_0^2 + f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}.$$

BASIS STEP: $P(0)$ is the proposition $f_0^2 = f_0 f_1$. It is true because $f_0^2 = 0^2 = 0$ and $f_0 f_1 = 0 \cdot 1 = 0$.

INDUCTIVE STEP: Suppose $P(k)$ is true. Then

$$f_0^2 + f_1^2 + f_2^2 + \dots + f_k^2 = f_k f_{k+1}.$$

We need to show that $P(k+1)$ is true: $f_0^2 + f_1^2 + f_2^2 + \cdots + f_{k+1}^2 = f_{k+1}f_{k+2}$. We take $P(k)$ and add f_{k+1}^2 to both sides of the equation, obtaining

$$\begin{aligned}(f_0^2 + f_1^2 + f_2^2 + \cdots + f_k^2) + f_{k+1}^2 &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1}f_{k+2} \quad (\text{using the Fibonacci sequence recurrence})\end{aligned}$$

Therefore $P(k+1)$ follows from $P(k)$.

Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all nonnegative integers n .

p.300, icon at Example 7

#1. Give a recursive definition for the set $S = \{4, 7, 10, 13, 16, 19, \dots\}$.

Solution:

The set can be written starting from 4 and adding 3 over and over.

BASIS STEP: $4 \in S$.

RECURSIVE STEP: $n \in S \rightarrow n + 3 \in S$.
