

Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 4.2—Strong Induction and Well-Ordering



— Page references correspond to locations of Extra Examples icons in the textbook.

p.285, icon at Example 2

#1. Consider an infinite checkerboard of squares, where all squares are white other than an initial set B_0 of n black squares; we call B_0 the initial generation of black squares. We define new generations of black squares recursively. Subsequent generations of black squares B_1, B_2, \dots are defined by the rule that a square is in B_k if and only if at least two of this square itself, the square directly above it, and the square directly to its right are in B_{k-1} . That is, a square on the checkerboard is in a new generation of black squares, if in the previous generation of black squares, there are more black squares than white squares among the square itself, the square above it, and the square to its right. Use strong induction to prove that $B_n = \emptyset$, that is, after n steps (where n is the number of initial black squares), no squares are black.

Solution:

BASIS STEP: If we start with one black square, there will be no squares in B_1 . This is the case for if there is only one black square in the initial generation, by the rule defining new generations, all squares in the following generation are white.

INDUCTIVE STEP: The inductive hypothesis is that if B_0 contains k squares where $k < n$, then $B_k = \emptyset$. That is, for the inductive hypothesis, we assume that if there are fewer than n squares in the initial set of black squares, then after k steps, no squares are black. Now consider an initial set B_0 of n black squares. Let R be the smallest rectangle that contains B_0 . (The borders of this rectangle are found using the uppermost, lowermost, rightmost, and leftmost black square, or squares, on the initial checkerboard.) Now let B'_0 consist of all squares in B_0 not in the bottom row of R . There are fewer than n squares in B'_0 , because by the definition of R , there is at least one black square in its bottom row. Furthermore, the colors of the squares in the bottom row of R are not used when determining which squares are black in the generations of sets of black squares that arise when we begin with B'_0 . Consequently, by the inductive hypothesis, after at most $n - 1$ steps, there are no black squares when we begin with B'_0 . Analogously, if we start with the set B''_0 consisting of all squares in B_0 not in the leftmost column of R , by the inductive hypothesis, after at most $n - 1$ steps, there are no black squares. Using what we have shown, it follows that when we start with B_0 , after $n - 1$ steps, no squares above the bottom row of R or the right of the leftmost column of R are black. Otherwise, there would have been at least one black square after $n - 1$ steps when we start with either B'_0 or B''_0 . Consequently, the only square that can be black after $n - 1$ steps is the lower lefthand corner of R . But at the n th step, this square must be white because it has no black neighbors. This completes the inductive step and the proof.

p.291, icon at Example 3

#1. Use the well-ordering property directly to show that if you can reach the first rung of an infinite ladder and if for every positive integer, if you can reach the n th rung, then you can reach the $(n + 1)$ st rung, then you can reach every rung.

Solution:

Suppose that there is at least one rung you cannot reach. By the well-ordering property, there is a least positive integer n such that you cannot reach the n th rung. We know that n cannot equal one, because by

hypothesis, you can reach the first rung. It follows that $n > 1$, and consequently, $n - 1$ is a positive integer. Because n is the least positive integer such that you cannot reach the n th rung, it follows that you can reach the $(n - 1)$ st rung. Furthermore, because you can reach the $(n - 1)$ st rung, by hypothesis, you can reach rung $(n - 1) + 1 = n$. This is a contradiction. It follows that you can reach all rungs of the ladder.
